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# Bessel Functions and Representation Theory, II
# Holomorphic Discrete Series and Metaplectic Representations

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This paper presents representation–theoretic applications of the general theory of operator-valued Bessel functions developed in the first paper of this series. Here, the concern is with the circle of ideas relating decomposition of the Fourier transform on $\mathbb{F}^{k\times n}$, $\mathbb{F}$ a real finite-dimensional division algebra and $k \geqslant 2n$, to metaplectic representations, holomorphic discrete series, and limits of holomorphic discrete series for the group of biholomorphic automorphisms of the Siegel upper half-plane in the complexification of $\mathbb{F}^{n\times n}$.

## 1. INTRODUCTION

This paper is devoted to representation–theoretic applications of the theory of Bessel functions begun in [1]. In particular, we are concerned with interrelationships between Bessel functions on matrix space and metaplectic representations, holomorphic discrete series, and limits of holomorphic discrete series for automorphism groups of certain homogeneous Siegel domains of type I.

To be more specific, let $\mathbb{F}$ be a real finite-dimensional division algebra, $H$ the generalized Siegel upper half-plane in the complexification of $\mathbb{F}^{n\times n}$, and $G = G(n, \mathbb{F})$ the group of biholomorphic automorphisms of $H$. Thus, $H = S + iP$ where $S$ consists of the self-adjoint elements of $\mathbb{F}^{n\times n}$ (i.e., elements $x$ such that $x = x'$, where $x' = \bar{x}^t$) and $P$ is the cone of positive-definite

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elements of $S$, and $G$ consists of all $2 \times 2$ matrices $g$ over $\mathbb{F}^{n \times n}$ such that $gpg' = p$ where $p = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In customary notation, $G$ is $Sp(n, \mathbb{R})$, $U(n, n)$, or $\mathcal{O}_*(4n)$ accordingly as $\mathbb{F}$ is real, complex, or quaternionic.

Now the group $G$ admits a family of *metaplectic representations* analogous to those arising in physics [11, 12] and number theory [13]. In fact, for each positive integer $k$ there exists a metaplectic representation $\mathscr{R}^{(k)}$ of $G$ which acts in the space $L^2(\mathbb{F}^{k \times n})$. Since these representations are in general highly reducible, it is natural to consider the following two problems.

(1) Describe the primary decomposition of $\mathscr{R}^{(k)}$ as explicitly as possible.

(2) Identify the irreducible constituents of $\mathscr{R}^{(k)}$ with known, or new, series of representations of $G$.

In this paper, we effect a complete solution to these problems when $k \geqslant 2n$. This requires, among other analytic considerations, a new realization of a portion of the holomorphic discrete series for $G$, the definition and basic properties of which (e.g., square-integrability and connection with metaplectic representations) rely heavily upon the *reduced Bessel functions* introduced in [1]. When $k = 2n$ and $\mathbb{F}$ is the real field, the construction of certain "limits" of holomorphic discrete series also enters the solution. We comment that in the real case with $k$ odd, $\mathscr{R}^{(k)}$ is actually a representation of the metaplectic group $Mp(n, \mathbb{R})$ which is a two-fold covering of $Sp(n, \mathbb{R})$.

We give a more detailed summary of this work. Set $X = \mathbb{F}^{k \times n}$ and let $U$ be the connected component of 1 in the compact group $\mathcal{O}(k, \mathbb{F}) = \{u \in \mathbb{F}^{k \times k}: uu' = 1\}$. Then the general theory of Bessel functions provides a partial solution to problem (1). Indeed, the fact that the Fourier transform on $L^2(X)$ decomposes under the left action of $U$ into Bessel transforms implies a decomposition

$$\mathscr{R}^{(k)} \cong \sum_{\lambda \in \hat{U}_X} \oplus \mathscr{R}(\cdot, \lambda, k). \tag{1.1}$$

Here, $\hat{U}_X$ is the subset of the dual object $\hat{U}$ of $U$ containing those representations $\lambda$ for which the $\lambda$th *order Bessel function*

$$J_\lambda(x, y) = \int_U e^{i \operatorname{Re} \operatorname{tr}(y' u x)} \lambda(u)^{-1} \, du \qquad (x, y \in X) \tag{1.2}$$

is not identically zero. The representation $\mathscr{R}(\cdot, \lambda, k)$ acts in the space of $\lambda$-covariant square-integrable functions from $X$ to the space of linear transformations on the representation space of $\lambda$, and the operator $\mathscr{R}(p, \lambda, k)$ is essentially the Bessel transform $\mathscr{J}_\lambda$ corresponding to $J_\lambda$. To show that $\mathscr{R}(\cdot, \lambda, k)$ is primary, and to identify it in relation to the holomorphic discrete series, is more difficult.

The holomorphic discrete series for $G$ [19] can be described as folows. $G$ acts transitively on $H$ by linear fractional transformations $z \circ g =$

$(zg_{12} + g_{22})^{-1}(zg_{11} + g_{21})$, and $H$ is an unbounded realization of the Hermitian symmetric space $G/K$, $K$ being the stability group of $i1 \in H$. Denote by $A = GL(n, \mathbb{F}^{\mathbb{C}})$ the group of invertible elements of the complexified algebra $(\mathbb{F}^{\mathbb{C}})^{n \times n}$. Then $K$ is isomorphic to the unitary subgroup of $A$ (cf., the proof of Theorem 6.4). For each irreducible holomorphic finite-dimensional (ihfd) representation $\pi$ of $A$ which is unitary on $K$, let $\mathfrak{H}_\pi$ consist of all holomorphic functions $F$ from $H$ to the space $\mathscr{H}_\pi$ of $\pi$, such that

$$\|F\|_\pi^2 = \int_H \|\pi(y)^{1/2} F(x + iy)\|^2 \, d_* z \tag{1.3}$$

where $z = x + iy \in H$ and $d_* z$ is a $G$-invariant measure on $H$. Of course, $\mathfrak{H}_\pi$ may be zero, but whenever it is nonzero the equation

$$(T(g, \pi)F)(z) = \pi(g_{12}' z + g_{22}')^{-1} F(z \circ g) \tag{1.4}$$

defines a unitary representation of $G$ in $\mathfrak{H}_\pi$ which is irreducible and square-integrable. These representations $T(\cdot, \pi)$ form the holomorphic discrete series.

One can construct a nonstandard realization $R(\cdot, \pi)$ of the representation $T(\cdot, \pi)$. Let $A_0$ be the connected component of 1 in $GL(n, \mathbb{F})$ and denote by $\mathscr{C}_0$ the "orthogonal" subgroup of matrices $\tau$ such that $\tau\tau' = 1$. Define $L_\pi^2(A_0)$ to be the space of all square-integrable Baire functions $f: A_0 \to \mathscr{H}_\pi$ which satisfy the covariance condition $f(\tau x) = \pi(\tau) f(x)$ for all $(\tau, x) \in \mathscr{C}_0 \times A_0$. Then the absolute convergence of the *generalized gamma integral,* defined by

$$\gamma_m(\pi) = \int_P e^{-2 \operatorname{tr} r} \pi(r) \, \Delta(r)^{-2m} \, dr \tag{1.5}$$

where $m = 1 + (n - 1)\nu/2$, $\nu$ is the real dimension of $\mathbb{F}$, and $\Delta$ is essentially the determinant (cf. (2.11)), is necessary and sufficient for the nonvanishing of $\mathfrak{H}_\pi$. Moreover, when (1.5) converges, the *Laplace transform* $\mathscr{T}_\pi: f \to F$, defined by

$$F(z) = c \int_{A_0} e^{i \operatorname{tr} za'a} \pi(a') \, f(a) \, d_* a \tag{1.6}$$

where $d_* a$ is a Haar measure on $A_0$ and $c$ is a constant depending only upon $n$, maps $L_\pi^2(A_0)$ one-to-one onto $\mathfrak{H}_\pi$. Thus, the representations $R(\cdot, \pi) = \mathscr{T}_\pi^{-1} T(\cdot, \pi) \mathscr{T}_\pi$ form a new realization of the holomorphic discrete series in the spaces $L_\pi^2(A_0)$. The norm on $L_\pi^2(A_0)$ which makes $\mathscr{T}_\pi$ unitary is given by

$$\|f\|_\pi^2 = \int_{A_0} \|\gamma_m(\pi)^{1/2} f(a)\|^2 \, d_* a. \tag{1.7}$$

We remark that the Laplace transform exists more generally for ifdh representations $\pi$ such that the integral

$$\gamma_0(\pi) = \int_P e^{-2\operatorname{tr} r}\pi(r)\, d_*r \tag{1.8}$$

converges absolutely, and the above construction leads in principle to certain "limits" of holomorphic discrete series. In terms of the *restricted highest weight* (Definition 3.2) $(l_1, l_2, ..., l_n)$ of $\pi$, the holomorphic discrete series exists for $l_n > 1 + (n-1)\nu$, and limits appear in the range $(n-1)\nu/2 < l_n \leqslant 1 + (n-1)\nu$.

Finally, we can identify the irreducible constituents of the primary decomposition (1.1) of $\mathscr{R}^{(k)}$. Each irreducible constituent appears with finite multiplicity and, when $\mathbb{F} \neq \mathbb{R}$ or $k > 2n$, is equivalent to a member of the holomorphic discrete series. In fact, the holomorphic discrete series representations which appear in the reduction of $\mathscr{R}^{(k)}$ are precisely those given by the ifdh representations $\pi^{(k)}$ of $A$ of the form

$$\pi^{(k)} = \delta_k \otimes \pi \tag{1.9}$$

where $\delta_k$ is a particular holomorphic character of $A$ which only depends upon $k$ (cf., Eqs. (5.11)–(5.13)), and $\pi$ is any irreducible finite-dimensional polynomial (ifdp) representation of $A$.

In more detail, the operator $R(p, \pi^{(k)})$ in the discrete series representation $R(\cdot, \pi^{(k)})$ is essentially the *reduced Bessel transform* $\mathscr{K}_\pi$ on $L_\pi^2(A_0)$ corresponding to the *reduced Bessel function* (cf., (2.6))

$$K_\pi(a) = \int_U e^{i \operatorname{Re} \operatorname{tr} a u_{11}'} \pi(u_{11} + iu_{21})\, du \qquad (a \in A_0) \tag{1.10}$$

for $\mathbb{F}^{k\times n}$. There is a correspondence $\pi \to \lambda = \lambda(\cdot, \pi) \in \hat{U}_X$ (cf., the discussion following Definition 2.2), such that the Bessel transforms $\mathscr{J}_\lambda$ and $\mathscr{K}_\pi$ (with multiplicity) are unitarily equivalent, and this fact has the consequence

$$\mathscr{R}(\cdot, \lambda, k) \cong d_\lambda R(\cdot, \pi^{(k)}) \qquad d_\lambda = \deg \lambda. \tag{1.11}$$

Thus, with the sole exception of the case in which $\mathbb{F} = \mathbb{R}$ and $k = 2n$,

$$\mathscr{R}^{(k)} \cong \sum_\pi \oplus\, d_\lambda R(\cdot, \pi^{(k)}) \qquad (\pi \text{ polynomial}) \tag{1.12}$$

is the primary decomposition of $\mathscr{R}^{(k)}$, and only holomorphic discrete series representations appear. The example in which $\mathscr{R}^{(2n)}$ acts in $L^2(\mathbb{R}^{2n\times n})$ is exceptional in that multiplicities are different,

$$\mathscr{R}^{(2n)} \simeq \sum_{\pi} \oplus\, 2d_\lambda R(\cdot, \pi^{(2n)}) \qquad (\pi \text{ polynomial}), \tag{1.13}$$

and not only do *all* holomorphic discrete series of $Sp(n, \mathbb{R})$ appear but also some limits of holomorphic discrete series.

Regarding related items in the literature, we mention the following papers. In [2] we studied the above problems when $\mathbb{F} = \mathbb{C}$ and $k = n$, and pointed out the appearance of both holomorphic discrete series and limits thereof in the primary decomposition of the representation $\mathscr{R}^{(n)}$ of $U(n, n)$. The results were extended in [15] to realize all the representations in the holomorphic discrete series for $U(n, n)$ in terms of generalized Bessel functions analogous to those in (1.10). Relevant here is the fact (cf., Example 8.8) that in the complex case there are holomorphic discrete series representations which fail to appear in the reduction of any metaplectic representation $\mathscr{R}^{(k)}$. For the real field, the above decompositions were originally obtained by Gelbart [16]. For general Hermitian symmetric spaces, Rossi and Vergne [7] have worked out the Paley–Weiner theorems relating the two realizations of holomorphic discrete series. Both Rossi and Vergne [17] and Wallach [18] have studied the "analytic continuation" of the holomorphic discrete series, which includes those limit representations corresponding to one-dimensional representations of the maximal compact subgroup. The full analytic continuation for the group $U(2, 2)$ appears in [21]. In [17] it is shown that such limits have realizations in Hardy spaces associated to boundary components of the symmetric space. For a certain boundary component, these are limits in the sense of Knapp and Okamoto [20]. In [18] it is shown that limits of holomorphic discrete series appear in metaplectic representations of $Sp(n, \mathbb{R})$ when $k < 2n$.

This paper is organized as follows. In Section 2 we review results in [1] on Bessel functions and Bessel transforms for matrix space. In Section 3 we bring together the information concerning the representations $\pi$ of $A$ needed to analyze the convergence of the generalized gamma function for $\mathbb{F}^{n\times n}$. Section 4 is ancillary to a proof of square-integrability of the representations $R(\cdot, \pi)$. There we examine the square-integrability of the reduced Bessel functions with respect to various measures. Section 5 concerns the Laplace transform and construction of, and analysis on, Hilbert spaces $\mathfrak{H}_{\pi,k}$ of holomorphic functions on $H$ analogous to the spaces $\mathfrak{H}_\pi$ above. These spaces contain multiplicity and are adapted for use later on in connection with $\mathscr{R}^{(k)}$. In Section 6 we construct primary representations $R(\cdot, \pi, k)$ and $T(\cdot, \pi, k)$ which are intertwined by a certain Laplace transform, $R(\cdot, \pi, k)$ being equivalent to $R(\cdot, \lambda, k)$ with $\lambda = \lambda(\cdot, \pi)$. We also compute the kernel function for the spaces $\mathfrak{H}_{\pi,k}$. The square-integrability of $R(\cdot, \pi, k)$ is proven in Section 7. There we also remark upon the restriction of holomorphic discrete series to a maximal parabolic subgroup. Finally, in Section 8 we review the construction of $\mathscr{R}^{(k)}$, describe its decomposition, and explain its relationship to holomorphic discrete series and its limits.

## 2. Bessel Functions on $\mathbb{F}^{k\times n}$

In this section we briefly review some results from [1] concerning Bessel functions and Hankel transforms on the space $\mathbb{F}^{k\times n}$, $k \geqslant 2n$. We refer to that paper for detailed proofs.

$\mathbb{F}$ denotes a real finite-dimensional division algebra (so $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), $X = \mathbb{F}^{k\times n}$ with $k \geqslant 2n$, $\mathscr{O}(k, \mathbb{F}) = \{u \in \mathbb{F}^{k\times k}: uu' = 1\}$, and $U$ denotes the connected component of 1 in the group $\mathscr{O}(k, \mathbb{F})$. Then $(U, X)$ is an orthogonal transformation group with respect to left matrix multiplication of $U$ on $X$. Let $\hat{U}$ denote a fixed set of representatives $\lambda$ for the equivalence classes of irreducible unitary representations of $U$, let $\mathscr{V}_\lambda$ be the space of $\lambda$, $d_\lambda = \deg \lambda$, and $\mathscr{L}_\lambda$ the space of linear transformations on $\mathscr{V}_\lambda$. Throughout, we use the inner product

$$(S \mid T) = \operatorname{tr}(T^*S) \tag{2.1}$$

for spaces of finite-dimensional linear transformations.

Definitions 2.1. Let $\lambda \in \hat{U}$. The function $J_\lambda: X \times X \to \mathscr{L}_\lambda$ defined by

$$J_\lambda(x, y) = \int_U e^{i \operatorname{Re} \operatorname{tr}(y'ux)} \lambda(u)^{-1}\, du \tag{2.2}$$

is called the $\lambda$th *order* (*generalized*) *Bessel function for* $(U, X)$. Here, $du$ is a normalized Haar measure on $U$ and $y' = \bar{y}^t$ where the bar denotes the usual conjugation on $\mathbb{F}$. The associated *Bessel*, or *Hankel*, *transform* $\mathscr{J}_\lambda$ acts on the Hilbert space $L_\lambda^2(X, \mathscr{L}_\lambda)$ of square-integrable $\lambda$-covariant Baire functions $\phi: X \to \mathscr{L}_\lambda$ by the formula

$$(\mathscr{J}_\lambda\phi)(x) = c_\pi \int_X J_\lambda(2x, y)\, \phi(y)\, dy \tag{2.3}$$

where $c_\pi = \pi^{-kn\nu/2}$, $\nu$ being the real dimension of $\mathbb{F}$.

Let $\mathscr{F}: f \to \hat{f}$ be the (unitary) Fourier transform on $L^2(X)$ defined by the formula

$$\hat{f}(x) = c_\pi \int_X e^{2i \operatorname{Re} \operatorname{tr}(y'x)} f(y)\, dy. \tag{2.4}$$

$\mathscr{J}_\lambda$ is unitary on $L_\lambda^2(X, \mathscr{L}_\lambda)$, and

$$\mathscr{F} \cong \sum_{\lambda \in \hat{U}_X} \oplus \mathscr{J}_\lambda \tag{2.5}$$

where $\hat{U}_X$ consists of all $\lambda$ in $\hat{U}$ such that $J_\lambda \neq 0$. In short, the Fourier transform decomposes under the action of $U$ into Bessel transforms.

Next, let $GL(n, \mathbb{F})$ be the group of invertible elements of the algebra $\mathbb{F}^{n \times n}$ and $A_0$ the component of 1 in $GL(n, \mathbb{F})$. Define $A = GL(n, \mathbb{F}^{\mathbf{C}})$ to be the group of invertible elements of the complexified algebra $(\mathbb{F}^{\mathbf{C}})^{n \times n}$. We regard $A_0$ as a real Lie subgroup of its complexification $A$. (In [1; Section 6] the symbol $Y$ was used for the group $A_0$.) Fix a complete set of representatives $\pi$ for the irreducible finite-dimensional polynomial (ifdp) representations of $A$. Let $\mathscr{H}_\pi$ be the space of $\pi$, and $\mathscr{L}_\pi$ the space of linear transformations on $\mathscr{H}_\pi$.

DEFINITION 2.2. For each ifdp representation $\pi$ of $A$ we define the *reduced Bessel function of order* $\pi$ for $(U, X)$ to be the mapping $K_\pi: A_0 \to \mathscr{L}_\pi$ given by

$$K_\pi(a) = \int_U e^{i \operatorname{Re} \operatorname{tr}(a u_{11}')} \pi(u_{11} + i u_{21})\, du \tag{2.6}$$

where we write a $k \times k$ matrix as $u = (u_{ij})$, $1 \leqslant i, j \leqslant 3$, relative to the block decomposition

$$\begin{array}{c|ccc} & n & n & s \\ \hline n & \cdot & \cdot & \cdot \\ n & \cdot & \cdot & \cdot \\ s & \cdot & \cdot & \cdot \end{array}, \qquad k = 2n + s. \tag{2.7}$$

The two kinds of Bessel functions, $J_\lambda$ and $K_\pi$, are related through Theorem 6.1 of [1] on invariants. Let

$$\mathbb{1} = \begin{pmatrix} 1_n \\ 0 \end{pmatrix} \in X$$

and $U_0$ the stability group of $\mathbb{1}$ in $U$ (cf., [1; Example 5.5]). Then $\lambda \in \hat{U}$ is in $\hat{U}_X$ if and only if the space

$$\mathscr{W}_\lambda = \{f \in \mathscr{V}_\lambda : \lambda(u_0) f = f \text{ for all } u_0 \in U_0\} \tag{2.8}$$

of $U_0$-invariants for $\lambda$ is nonzero. Moreover, there is a one-to-one mapping $\pi \to \lambda = \lambda(\cdot, \pi)$ from the ifdp representations of $A$ into $\hat{U}_X$, and a vector space isomorphism $T_0$ from $\mathscr{W}_\lambda$ onto $\mathscr{H}_\pi$ such that for all $a_1, a_2 \in A_0$

$$K_\pi(a_1 a_2') = T_0 J_\lambda(a_1, a_2)\, T_0^{-1}. \tag{2.9}$$

(Here, $a \in A_0$ is embedded in $X$ as $\mathbb{1}a$.) When $\mathbb{F} \neq \mathbb{R}$ or $k > 2n$, the mapping $\pi \to \lambda(\cdot, \pi)$ is a bijection onto $\tilde{U}_X$. The exceptional example $\mathbb{R}^{2n\times n}$ will be treated in full in Section 8.

At this point we need to recall formulas for certain measures from Example 5.5 of [1]. Let $P$ be the cone of positive-definite self-adjoint elements of $\mathbb{F}^{n\times n}$, let $det$ be the reduced norm on $\mathbb{F}^{n\times n}$, and set $\Delta(x) = (det\, x)^j$ with $j = 1$ when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $j = \frac{1}{2}$ when $\mathbb{F} = \mathbb{H}$. (In the quaternionic case $det\, x \geqslant 0$ for all $x$, so the square root is well defined.) Fix Lebesgue measures $da$ and $dr$ on $\mathbb{F}^{n\times n}$ and $P$, respectively. Then

$$d_*a = |\Delta(a)|^{-\nu n}\, da \tag{2.10}$$

defines Haar measure on $A_0$, and

$$d_*r = \Delta(r)^{-m}\, dr, \qquad m = \tfrac{1}{2}\{(n-1)\nu + 2\} \tag{2.11}$$

defines a measure on $P$ invariant under the action $r \to ara'$ of $A_0$. Let $\mathcal{O}_0$ be the component of 1 in $\mathcal{O}(n, \mathbb{F})$. (In [1; Section 6] $\mathcal{O}_0$ was denoted by $V$.) In terms of polar coordinates $a = \tau r^{1/2}$ ($\tau \in \mathcal{O}_0$, $r = a'a \in P$) for $A_0$

$$d_*a = \beta\, d\tau\, d_*r \tag{2.12}$$

for some positive constant $\beta = \beta_n$, where $d\tau$ is normalized Haar measure on $\mathcal{O}_0$. Finally, let $dx$ be Lebesgue measure on $X$ and $x = u\mathbb{1}r^{1/2}$ polar coordinates on $X$ ($u \in U$, $r = x'x$). Then

$$dx = \beta\, du\, \Delta(r)^{k\nu/2}\, d_*r \tag{2.13}$$

where $\beta = \beta_{k,n}$ is a positive constant. (As a matter of convenience we shall use the symbol $\beta$ for a generic constant, the value of which may change with the context.)

Let $\mathscr{L}_{\lambda,\pi}$ be the space of linear transformations from $\mathscr{V}_\lambda$ to $\mathscr{H}_\pi$, and for a fixed linear transformation $\mathscr{Q}\colon \mathscr{H}_\pi \to \mathscr{W}_\lambda$ let $L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, \mathscr{Q})$ be the space of square-integrable Baire functions $f\colon A_0 \to \mathscr{L}_{\lambda,\pi}$ such that

$$f(\tau a) = \pi(\tau) f(a), \qquad (\tau, a) \in \mathcal{O}_0 \times A_0, \tag{2.14}$$

in which the inner product is defined by the formula

$$(f \mid g) = d_\lambda \int_{A_0} \operatorname{tr}(g(a)^*\, \mathscr{Q}^*\mathscr{Q} f(a))\, d_*a \tag{2.15}$$

for $f, g \in L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, \mathscr{Q})$. Then there exists a unitary operator $\Phi$ from $L_\lambda^2(X, \mathscr{L}_\lambda)$ onto $L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0^{-1})$ such that

$$\mathscr{K}_\pi = \Phi \mathscr{J}_\lambda \Phi^{-1} \tag{2.16}$$

is given by the formula

$$(\mathscr{K}_\pi f)(a) = \beta c_\pi \int_{A_0} K_\pi(2ab') \mid \Delta(ab')|^{k\nu/2} f(b)\, d_* b \tag{2.21}$$

whenever $\Delta^{k\nu/2} f$ is integrable, $\beta$ being the positive constant $\beta_{k,n}\beta_n^{-1}$ related to those in (2.12) and (2.13). Thus, the Bessel transform $\mathscr{J}_\lambda$ is equivalent to the *reduced Bessel transform* $\mathscr{K}_\pi$.

Finally, we derive a variant of Theorem 6.6 of [1]. Observe from (2.9) and (2.2) that

$$K_\pi(a)^* = (T_0 T_0^*)^{-1} K_\pi(-a')\, T_0 T_0^* \tag{2.18}$$

for all $a \in A_0$, and from (6.7) in [1] that

$$T_0 T_0^* \pi(\tau) = \pi(\tau)\, T_0 T_0^* \tag{2.19}$$

for all $\tau \in C_0$. In particular, if the restriction $\tau \to \pi(\tau)$ of $\pi$ to $C_0$ is irreducible then $T_0$ is essentially unitary and (2.18) takes the simple form $K_\pi(a)^* = K_\pi(-a')$. In any case, we see that the mapping $f(a) \to (T_0 T_0^*)^{-1} f(a)$ is unitary from $L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0^{-1})$ to $L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0^*)$, and by (2.18) the operator on the latter space corresponding to $\mathscr{K}_\pi$ is

$$(\tilde{\mathscr{K}}_\pi f)(a) = \beta c_\pi \int_{A_0} K_\pi(-2ba')^* \mid \Delta(ab')|^{k\nu/2} f(b)\, d_* b. \tag{2.20}$$

It will be technically convenient later to work with the inverse of $\tilde{\mathscr{K}}_\pi$. This amounts to replacing the Fourier transform by its inverse. The formula, for $f \in L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0^*)$ such that $\Delta^{k\nu/2} f$ is integrable, is

$$(\tilde{\mathscr{K}}_\pi^{-1} f)(a) = \beta c_\pi \int_{A_0} K_\pi(2ba')^* \mid \Delta(ab')|^{k\nu/2} f(b)\, d_* b \quad . \tag{2.21}$$

where $\beta$ is as in (2.17). Now let $H = S + iP$ be the Siegel upper half-plane in $(\mathbb{F}^{\mathbb{C}})^{n\times n}$, $S$ being the space of self-adjoint elements $x = x'$ of $\mathbb{F}^{n\times n}$. Taking adjoints on each side of formula (6.25) of [1] and making the change of variables $z \to -z^*$ in $H$ (cf., the second paragraph of Section 6), we obtain

$$\int_P e^{i\,\mathrm{tr}\,zr} \pi(r^{1/2})\, K_\pi(2ar^{1/2})^*\, \Delta(r)^{k\nu/2}\, d_* r$$

$$= \beta \Delta(iz^{-1})^{k\nu/2}\, \pi(z^{-1}a')\, e^{-i\,\mathrm{tr}(a'az^{-1})} \tag{2.22}$$

where, from Example 5.10 and Theorem 6.6 of [1], $\beta = \beta_{k,n}^{-1} c_\pi^{-1}$. Here, $\Delta(iz^{-1})^{k\nu/2}$ is defined by analytic continuation on the "right half-plane" $-iH$. Note that we have used the property that $\pi(a)^* = \pi(a^*)$ for all $a \in A$. This characteristic of $\pi$ is equivalent to the property that its restriction to the maximal compact (unitary) subgroup of $A$ be a *unitary* representation (cf., Section 3). Using formula (2.12), we can put (2.22) into the alternative form

$$\beta c_\pi \int_{A_0} e^{i \operatorname{tr} zb'b} \pi(b')\, K_\pi(2ab')^* \mid \Delta(b)|^{k\nu}\, d_*b$$

$$= e^{-i \operatorname{tr} z^{-1}a'a} \Delta(iz^{-1})^{k\nu/2}\, \pi(z^{-1}a') \tag{2.23}$$

where $\beta$ is the same constant as in (2.21). Formula (2.23) is central to the results of this paper.

## 3. The Gamma Function for $\mathbb{F}^{n\times n}$

In this section we study a generalization to $\mathbb{F}^{n\times n}$ of the classical gamma function.

Let $\pi$ denote the generic irreducible holomorphic finite-dimensional (ihfd) representation of $A$, and let $\mathscr{H}_\pi$ be the space of $\pi$. Let $\tilde{\pi}$ denote the restriction of $\pi$ to the subgroup $A_0$. Since $A_0$ is a real form of $A$, $\tilde{\pi}$ is an irreducible (real-analytic) representation of $A_0$. As before let $H = \{z = x + iy\colon x \in S, y \in P\}$ be the Siegel upper half-plane in $(\mathbb{F}^{\mathbf{C}})^{n\times n}$.

Definition 3.1. For $z \in H$ and ihfd representation $\pi$ of $A$, let

$$\Gamma(z, \pi) = \int_P e^{i \operatorname{tr} zr} \pi(r)\, d_*r \tag{3.1}$$

whenever the integral converges absolutely; i.e.,

$$\int_P e^{-\operatorname{tr} yr}\, \| \pi(r)\|\, d_*r < \infty,$$

the norm being given by (2.1). The mapping $(z, \pi) \to \Gamma(z, \pi)$ is called the (generalized) *gamma function for* $\mathbb{F}^{n\times n}$.

In order to investigate the convergence of (3.1) we need more information about the representations $\pi$. For this purpose we realize $\pi$ as "induced." We refer to [4, 5], or [6] for the general theory.

Let $C$ be a Cartan subgroup of $A$ and $N$ and $V$ opposite maximal unipotent subgroups normalized by $C$. One can realize $(\mathbb{F}^{\mathbf{C}})^{n\times n}$, and it is convenient to do so, in a space of square complex matrices in such a way that $C$, $CN$, and $CV$ are the diagonal, lower triangular, and upper triangular subgroups of $A$, respectively. The ihfd representations of $A$ are indexed by a certain collection $D$ of holomorphic characters of $C$ called ***highest weights***. Each highest weight $\sigma$ can be written uniquely as

$$\sigma(c) = \Delta_\sigma(c)\, \sigma_p(c), \qquad c \in C \tag{3.2}$$

where $\sigma_p$ is a polynomial character of $C$ which is itself a highest weight and $\Delta_\sigma$ is a rational character of $A$ (an integral power of the determinant). The corresponding ihfd representation $\pi = \pi_\sigma$ of $A$ acts in a space $\mathscr{P}_\sigma$ of polynomial functions on $A$ by the formula

$$(\pi_\sigma(a)f)(x) = \Delta_\sigma(a) f(xa). \tag{3.3}$$

The space $\mathscr{P}_\sigma$ has the following explicit description. Let $B = CN$ be the lower triangular subgroup of $A$, and extend $\sigma_p$ to a polynomial character of $B$ which restricts to the identity on $N$. Then $\mathscr{P}_\sigma$ is composed of all polynomial functions $f\colon A \to \mathbb{C}$ such that

$$f(bx) = \sigma_p(b) f(x) \tag{3.4}$$

for $(b, x) \in B \times A$. $\mathscr{P}_\sigma$ is nonzero, finite-dimensional, and right invariant. Let $K$ denote the maximal compact (i.e., unitary) subgroup of $A$. Since $A = BK$, the equation

$$(f \mid g) = \int_K f(k)\, \overline{g(k)}\, dk \tag{3.5}$$

for $f, g \in \mathscr{P}_\sigma$ defines an inner product on $\mathscr{P}_\sigma$, and with respect to this inner product $\pi(k)$ is unitary for each $k \in K$. More generally,

$$\pi_\sigma(a)^* = \pi_\sigma(a^*) \tag{3.6}$$

for all $a \in A$, where $*$ denotes the conjugate-linear extension to $(\mathbb{F}^{\mathbb{C}})^{n\times n}$ of the involution in $\mathbb{F}^{n\times n}$.

Now, a ***highest weight vector*** for $\pi_\sigma$ is a nonzero vector $f_\sigma \in \mathscr{P}_\sigma$ such that

$$\pi_\sigma(cv) f_\sigma = \sigma(c) f_\sigma \tag{3.7}$$

for all $(c, v) \in C \times V$. Up to scalar multiples, the highest weight vector is unique.

The set $D$ of highest weights parametrizes the collection of all ihfd representations of $A$. For if $\pi$ is any such representation of $A$, then there exists a unique $\sigma \in D$ such that $\pi$ is equivalent to $\pi_\sigma$. Throughout the paper, we use the notation $\pi = \pi_\sigma$, $\sigma \in D$, for the generic ihfd representation of $A$, and hence $\mathscr{H}_\pi = \mathscr{P}_\sigma$.

Those representations $\pi$ which are polynomial can be described as follows. Let $\Delta_\sigma$ be the rational character $\Delta_\sigma$ of $A$ in (3.2), and let

$$D^p = \{\sigma \in D\colon \Delta_\sigma \text{ is polynomial}\}. \tag{3.8}$$

By (3.3), $D^p$ indexes the ifdp representations of $A$. These are the representations of $A$ that appeared in Section 2.

Next, we consider the restriction $\tilde{\pi}$ of the ihfd representation $\pi$ to the real subgroup $A_0$ of $A$. This is an irreducible real-analytic representation of $A_0$ in the space $\mathscr{P}_\sigma$. Let $C_0^+ = C \cap P$, a Cartan subgroup of $A_0$. The elements $c$ of $C_0^+$ are parametrized by real numbers $c_1, \ldots, c_n$, in terms of which the restriction $\sigma_0$ of $\sigma$ to $C_0^+$ has the form

$$\sigma_0(c) = c_1^{l_1} c_2^{l_2} \cdots c_n^{l_n} \tag{3.9}$$

where $l_1, \ldots, l_n$ are integers such that

$$l_1 \geqslant \cdots \geqslant l_n . \tag{3.10}$$

DEFINITION 3.2. The integers $l_1, \ldots, l_n$ are uniquely determined by (3.9) and (3.10). We call $\sigma_0 = (l_1, \ldots, l_n)$ as above, the ***restricted highest weight*** of $\pi = \pi_\sigma$ and we set

$$\omega(\pi) = l_n . \tag{3.11}$$

To make the above representation theory more explicit we briefly examine the real, complex, and quaternionic cases individually.

EXAMPLE 3.3. If $\mathbb{F}$ is the real field, then $A = GL(n, \mathbb{C})$, $A_0 = \{x \in GL(n, \mathbb{R}) : \det x > 0\}$, $C$ is the diagonal subgroup of $A$, $B$ is the full lower triangular subgroup, and as characters of $B$

$$\sigma(b) = b_{11}^{s_1} b_{22}^{s_2} \cdots b_{nn}^{s_n} , \tag{3.12}$$

$$\sigma_p(b) = b_{11}^{s_1 - s_n} b_{22}^{s_2 - s_n} \cdots b_{n-1,n-1}^{s_{n-1} - s_n} , \tag{3.13}$$

and

$$\Delta_\sigma(b) = (\det b)^{s_n} \tag{3.14}$$

where $s_1, \ldots, s_n$ are integers such that

$$s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n . \tag{3.15}$$

In this case, $D = D_{\mathbb{R}}$ may be identified with the set of $n$-tuples $\sigma = (s_1, \ldots, s_n)$ of integers satisfying (3.15) and $D^p = \{s \in D : s_n \geqslant 0\}$. For $x \in \mathbb{C}^{n \times n}$ let

$$\Delta_k(x) = \det \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix}, \qquad 1 \leqslant k \leqslant n \tag{3.16}$$

and $f_\sigma$: $\mathbb{C}^{n\times n} \to \mathbb{C}$ the polynomial function

$$f_\sigma(x) = \Delta_1(x)^{s_1-s_2}\, \Delta_2(x)^{s_2-s_3} \cdots \Delta_{n-1}(x)^{s_{n-1}-s_n}. \tag{3.17}$$

Then the right translates of $f_\sigma$ under $GL(n, \mathbb{C})$ span the finite-dimensional space $\mathscr{P}_\sigma$, $\pi = \pi_\sigma$ is given by

$$(\pi(a)f)(x) = (\det a)^{s_n} f(xa) \tag{3.18}$$

for $a \in GL(n, \mathbb{C})$, and $f_\sigma$ is a highest weight vector. In this case, comparison of (3.12) and (3.9) shows that $l_i = s_i$ for $i = 1,\ldots, n$, and $\omega(\pi) = s_n$.

EXAMPLE 3.4. In the complex case we realize $A$ as $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, and $A_0$ as the real subgroup of elements $(x, \bar{x})$, $x \in GL(n, \mathbb{C})$. To be precise, let $M = \mathbb{C}^{n\times n} \oplus \mathbb{C}^{n\times n}$ as a complex algebra, and $\psi$ the mapping from $\mathbb{C}^{n\times n}$ into $M$ given by

$$\psi(x) = (x, \bar{x}), \qquad x \in \mathbb{C}^{n\times n}. \tag{3.19}$$

Then $\psi$ is an isomorphism of $\mathbb{C}^{n\times n}$, as a real algebra, onto a real subalgebra $M_0$ of $M$ such that $M_0 \cap iM_0 = \{0\}$ and $M = M_0 + iM_0$. Thus, $M$ is the complexification of $M_0$, or equivalently of $\mathbb{C}^{n\times n}$. In the complex case, therefore, we write elements of $A$ as

$$a = (a_1, a_2), \qquad a_1, a_2 \in GL(n, \mathbb{C}), \tag{3.20}$$

and the elements of $A_0$ as in (3.19). Then with the notation of Example 3.3, $D = D_{\mathbb{C}}$ can be identified with pairs of elements of $D_{\mathbb{R}}$; i.e., $\sigma = (\sigma^{(1)}, \sigma^{(2)})$ where $\sigma^{(1)} = (s_1, \ldots, s_n)$, $\sigma^{(2)} = (t_1, \ldots, t_n)$ satisfy the inequalities

$$s_1 \geqslant \cdots \geqslant s_n \qquad \text{and} \qquad t_1 \geqslant \cdots \geqslant t_n. \tag{3.21}$$

For $\sigma \in D$, the space $\mathscr{P}_\sigma$ consists of all polynomial functions

$$f\colon GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \to \mathbb{C} \text{ such that}$$

$$f(b_1x_1, b_2x_2) = \sigma_p^{(1)}(b_1)\, \sigma_p^{(2)}(b_2)\, f(x_1, x_2) \tag{3.22}$$

for $(b_1, b_2) \in B \times B$, $(x_1, x_2) \in A$; where $B$, $\sigma_p^{(1)}$, and $\sigma_p^{(2)}$ are as in Example 3.3. The representation $\pi = \pi_\sigma$ acts in $\mathscr{P}_\sigma$ by the formula

$$(\pi(a_1, a_2)f)(x_1, x_2) = (\det a_1)^{s_n} (\det a_2)^{t_n} f(x_1a_1, x_2a_2). \tag{3.23}$$

In particular, in the complex case

$$\Delta_\sigma(a) = (\det a_1)^{s_n} (\det a_2)^{t_n}, \qquad a = (a_1, a_2) \in A, \tag{3.24}$$

so $D^p = \{\sigma \in D\colon s_n \geqslant 0 \text{ and } t_n \geqslant 0\}$.

Now, $D = D_{\mathbb{C}}$ also serves to parametrize the irreducible real-analytic finite-dimensional representations of $GL(n, \mathbb{C})$. These are the restrictions $\tilde{\pi}$ to $A_0$ of the representations $\pi = \pi_\sigma$ of $A$; i.e., for $f \in \mathscr{P}_\sigma$ and $a \in GL(n, \mathbb{C})$

$$(\tilde{\pi}(a)f)(x) = (\det a)^{s_n} \overline{(\det \bar{a})}^{t_n} f(x_1 a, x_2 \bar{a}). \tag{3.25}$$

In this case, the subgroup $C_0^+$ of $A_0$ consists of the elements $c_0 = (x, x)$ where $x = \operatorname{diag}(c_1, \ldots, c_n)$ with $c_j > 0$. Hence, for $\sigma \in D$

$$\sigma(c_0) = c_1^{s_1+t_1} c_2^{s_2+t_2} \cdots c_n^{s_n+t_n} \tag{3.26}$$

so the restricted highest weight of $\pi$ is $\sigma_0 = (s_1 + t_1, \ldots, s_n + t_n)$ and $\omega(\pi_\sigma) = s_n + t_n$.

EXAMPLE 3.5. Let $\mathbb{F} = \mathbb{H}$, the quaternion algebra, realized as $2 \times 2$ complex matrices of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$ for $z, w \in \mathbb{C}$. Then $A_0 = GL(n, \mathbb{H})$ is a real group of $2n \times 2n$ complex matrices, and $A = GL(2n, \mathbb{C})$. Therefore, in this case the highest weights can be taken to be $2n$-tuples $\sigma = (s_1, \ldots, s_{2n})$ of integers such that

$$s_1 \geqslant \cdots \geqslant s_{2n}, \tag{3.27}$$

and the representations $\pi = \pi_\sigma$ of $A$ are given by (3.18) with $n$ replaced by $2n$. The subgroup $C_0^+$ of $GL(n, \mathbb{H})$ consists of $2n \times 2n$ complex matrices of the form $c = \operatorname{diag}(c_1, c_1; c_2, c_2; \ldots; c_n, c_n)$ with $c_j > 0$. By (3.12) and (3.27), for $\sigma \in D = D_{\mathbb{H}}$ and $c \in C_0^+$

$$\sigma(c) = c_1^{s_1+s_2} c_2^{s_3+s_4} \cdots c_n^{s_{2n-1}+s_{2n}} \tag{3.28}$$

so by (3.9) the restricted highest weight of $\sigma$ in the quaternionic case is $\sigma_0 = (s_1 + s_2, s_3 + s_4, \ldots, s_{2n-1} + s_{2n})$ and $\omega(\pi) = s_{2n-1} + s_{2n}$. Clearly, $D^p = \{\sigma \in D: s_{2n} \geqslant 0\}$.

We return to the general context, and use the preceding representation theory to describe the convergence of the gamma function. We extend formula (3.1) in the following manner. Let $\alpha \in \mathbb{C}$, $z \in H$, and $\pi$ an ihfd representation of $A$. Set

$$\Gamma(z, \alpha, \pi) = \int_P e^{i \operatorname{tr} zr} \pi(r)\, \Delta(r)^{-m+\alpha\nu/2}\, dr \tag{3.29}$$

($m$ is given by (2.11)), provided the integral converges absolutely. When $\alpha$ and $\alpha\nu/2$ are integers (3.29) reduces to (3.1) for an appropriate holomorphic representation of $A$.

Let $\mathscr{E}$ be the upper triangular subgroup of $A_0$ with positive diagonal entries, and let $A_0 = \mathcal{O}_0\mathcal{E}$ be the ***Iwasawa decomposition*** of $A_0$. From [2; Section 4] the corresponding decomposition of Haar measure is

$$d_*a = d\tau\, db \tag{3.30}$$

($a = \tau b \in A_0$ with $\tau \in \mathcal{O}_0$, $b \in \mathcal{E}$) for suitably normalized right Haar measure $db$ on $\mathcal{E}$.

THEOREM 3.6. *Let $\alpha \in \mathbb{C}$, $\sigma \in D$, $\pi = \pi_\sigma$, and $z \in H$. Then the integral defining $\Gamma(z, \alpha, \pi)$ converges absolutely if and only if*

$$\omega(\pi) > (n - \operatorname{Re} \alpha - 1)\nu/2. \tag{3.31}$$

*When this is the case,*

$$\Gamma(aza', \alpha, \pi) = \Delta(a'a)^{-\alpha\nu/2}\, \pi(a')^{-1}\, \Gamma(z, \alpha, \pi)\, \pi(a)^{-1} \tag{3.32}$$

*for all $z \in H$ and $a \in A_0$.*

*Proof.* Formula (3.32) follows from the invariance of the measure $d_*r = \Delta(r)^{-m}\, dr$ under the transformation $r \to a'^{-1}ra^{-1}$. We turn to the absolute convergence of (3.29). For this we may assume $\alpha$ is real. By (2.1) and (3.6), $\operatorname{tr}(\pi(r)) = \| \pi(r^{1/2})\|^2$. Since $\operatorname{tr} \pi(r) \geqslant \| \pi(r)\|$, it therefore suffices to consider the convergence of

$$\int_P | e^{i \operatorname{tr} zr} |\, \| \pi(r^{1/2})\|^2\, \Delta(r)^{\alpha\nu/2}\, d_*r$$

or equivalently

$$\int_P e^{-\operatorname{tr} yr}\, \| \pi(r^{1/2})f \|^2\, \Delta(r)^{\alpha\nu/2}\, d_*r$$

for $y \in P$ and $f \in \mathcal{H}_\pi$. By (2.12) this last integral is $\beta^{-1}I(y, f)$ where

$$I(y, f) = \int_{A_0} e^{-\operatorname{tr} ya'a}\, \| \pi(a)f \|^2\, \Delta(a'a)^{\alpha\nu/2}\, d_*a \tag{3.33}$$

for $y \in P$ and $f \in \mathcal{H}_\pi$. To complete the proof of the theorem we must show that $I(y, f) < \infty$ for all $y$ and $f$ if and only if $\omega(\pi) > (n - 1 - \alpha)\nu/2$.

Following Godement [3; Exposé 5] (cf., [7] as well), we observe that $I(y, f) < \infty$ for all $y \in P$ and $f \in \mathcal{H}_\pi$ if and only if there exists one nonzero function $f_0$ in $\mathcal{H}_\pi$ such that $I(y, f_0) < \infty$. We give a sketch of the proof. For $y \in P$ let $C$ and $D$ be positive constants which bound the spectrum of $y$ from above and below, respectively. Then $e^{-C \operatorname{tr} a'a} \leqslant e^{-\operatorname{tr} ya'a} \leqslant e^{-D \operatorname{tr} a'a}$ for all $a$ in $A_0$, from which one sees that $I(y, f) < \infty$ if and only if $I(1, f) < \infty$. Let $\mathcal{S}$ denote the set of all $f \in \mathcal{H}_\pi$ such that $I(1, f) < \infty$. Applying Minkowski's inequality to $I(1, f)^{1/2}$, we see that $\mathcal{S}$ is a subspace of $\mathcal{H}_\pi$. From the translationinvariance of Haar measure it follows that $\mathscr{S}$ is invariant under $\pi(a)$ for all $a$ in $A_0$, and since $\tilde{\pi}$ is irreducible we conclude that either $\mathscr{S}$ is the zero space or $\mathscr{S} = \mathscr{H}_\pi$.

Thus, if suffices to compute $I(1, f_0)$ for just one nonzero element of $\mathscr{H}_\pi$. To this end, let $f_0$ be a highest weight vector in $\mathscr{H}_\pi = \mathscr{P}_\sigma$. Then by (3.7), for $c \in C_0^+$ and $v \in V_0$, the upper triangular unipotent subgroup of $A_0$,

$$\pi(cv) f_0 = \sigma_0(c) f_0 \tag{3.34}$$

where $\sigma_0$ is the restricted highest weight of $\pi = \pi_\sigma$ given by Definition 3.2. By (3.30), (3.33), and (3.34)

$$\begin{aligned} I(1, f_0) &= \int_{\mathscr{E}} e^{-\operatorname{tr} b'b} \| \pi(b) f_0 \|^2 \Delta(b'b)^{\alpha\nu/2}\, db \\ &= \int_{V_0} \int_{C_0^+} e^{-\operatorname{tr}(c'v'vc)} \| \pi(vc) f_0 \|^2 \Delta(c'v'vc)^{\alpha\nu/2}\, dv\, dc \\ &= \int_{C_0^+} \sigma_0(c)^2 \Delta(c)^{\alpha\nu} \left( \int_{V_0} e^{-\operatorname{tr}(c'v'vc)}\, dv \right) dc \end{aligned}$$

where $\mathscr{E} = V_0 C_0^+$ is the semidirect product of $V_0$ with $C_0^+$, $b = vc$ with $v \in V_0$ and $c \in C_0^+$, and $db = dv\, dc$ where $dc$ is the standard invariant measure on $C_0^+$ and $dv$ is suitably normalized Haar measure on $V_0$. Let $W$ be the vector space of all upper triangular nilpotent $n \times n$ matrices $w$ over $\mathbb{F}$; i.e., $w = (w_{ij})$ where $w_{ij} = 0$ for $j \leqslant i$, and set $v = 1 + w$ for $v \in V_0$. Since $w$ is nilpotent and $c$ is diagonal

$$\int_{V_0} e^{-\operatorname{tr}(c'v'vc)}\, dv = e^{-\operatorname{tr}(c'c)} \int_W e^{-\operatorname{tr}(c'w'wc)}\, dw.$$

The Jacobian of the linear transformation $w \to wc^{-1}$ of $W$ is the function $\mu(c)$ given by

$$\mu(c) = (c_2 c_3{}^2 \cdots c_n^{n-1})^{-1}; \tag{3.35}$$

so the above integral is

$$\int_{V_0} e^{-\operatorname{tr}(c'v'vc)}\, dv = K \mu(c)^\nu\, e^{-\operatorname{tr}(c'c)}$$

where $K = \int_W e^{-\operatorname{tr}(w'w)}\, dw > 0$. It now follows that

$$\begin{aligned} I(1, f_0) &= K \int_{C_0^+} \sigma_0(c)^2 \Delta(c)^{\alpha\nu} \mu(c)^\nu e^{-\operatorname{tr}(c'c)}\, dc \\ &= K \prod_{j=1}^n \int_0^\infty c_j^{2\xi(j)} \exp(-c_j{}^2)\, c_j^{-1} dc_j \end{aligned}$$

where

$$\xi(j) = l_j + (\nu/2)(\alpha - j + 1).$$

Clearly, $I(1, f_0) < \infty$ if and only if

$$\operatorname{Re}(\xi(j)) > 0 \qquad \text{for} \quad j = 1, 2, \dots, n \tag{3.36}$$

in which case

$$I(1, f_0) = 2^{-n} K \prod_{j=1}^{n} \Gamma(\xi(j)).$$

To complete the proof of this theorem we need only show that conditions (3.36) and (3.31) coincide. On the one hand, the special case $j = n$ of (3.36) is precisely (3.31). Conversely, write (3.36) as $\zeta(j) > (j - 1)\nu/2$ where $\zeta(j) = l_j + \alpha\nu/2$, and recall that $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_n$. Hence, since $\zeta(j) = \zeta(j + 1) + (l_j - l_{j+1})$, we see that $\zeta(j) \geqslant \zeta(j + 1)$ for $j = 1, \dots, n - 1$. As (3.31) is precisely the condition $\zeta(n) > (n - 1)\nu/2$, (3.31) implies that $\zeta(j) \geqslant \zeta(j + 1) \geqslant \cdots \zeta(n) > (n - 1)\nu/2 > (j - 1)\nu/2$ for all $j = 1, \dots, n - 1$. Thus, (3.31) implies (3.36), and the proof is complete.

COROLLARY 3.7. *Let $\pi = \pi_\iota$ and $\alpha$ a real number such that $\omega(\pi) > (n - \alpha - 1)\nu/2$. Then*

$$N_{\pi,\alpha} = \Gamma(2i, \alpha, \pi) \tag{3.37}$$

*is a positive definite linear transformation on $\mathscr{H}_\pi$. Moreover,*

$$\pi(\tau)\, N_{\pi,\alpha} = N_{\pi,\alpha} \pi(\tau) \tag{3.38}$$

*for all $\tau \in \mathcal{O}_0$. In particular, if the representation $\tau \to \pi(\tau)$ of $\mathcal{O}_0$ is irreducible, then $N_{\pi,\alpha}$ is scalar.*

*Proof.* Formula (3.38) is immediate from (3.32). That $N_{\pi,\alpha}$ is positive is clear from the integral defining it.

The case in which $\pi \mid \mathcal{O}_0$ is irreducible is technically convenient. For then it is easy to construct new Hilbert spaces of holomorphic functions on $H$ which realize "limits" of holomorphic discrete series representations. This condition of irreducibility obtains not only when $\pi$ is one-dimensional, but also in Examples 3.3 and 3.5 for highest weights of the form $(l + 1, l, l, \dots, l)$ and in Example 3.4 for highest weights of the form $(s_1, \dots, s_n; l, \dots, l)$ and $(l, \dots, l; t_1, \dots, t_n)$.

## 4. SQUARE-INTEGRABILITY OF THE BESSEL FUNCTIONS

In this section we examine the square-integrability of $K_\pi$ with respect to various measures. We begin with a Laplace transform formula.

THEOREM 4.1. *Let $\pi$ be an* ihfd *representation of $A$, $\alpha$ any real number such that $\omega(\pi) > (n - \alpha - 1)\nu/2$, $\mathscr{V}$ any finite-dimensional complex Hilbert space, $\mathscr{L} = \mathscr{L}(\mathscr{V}, \mathscr{H}_\pi)$ the space of linear transformations from $\mathscr{V}$ to $\mathscr{H}_\pi$, and let $m$ be as in* (2.11). *Suppose that $\phi: P \to \mathscr{L}$ is a Baire function such that the integral*

$$\Phi(z) = (2\pi)^{-nm/2} \int_P e^{i \operatorname{tr} zr} \pi(r^{1/2}) \phi(r)\, dr \tag{4.1}$$

*is absolutely convergent for all $z \in H$. Then $\Phi$ is holomorphic on $H$ and*

$$\int_H \operatorname{tr}(\Phi(x+iy)^* \pi(y) \Phi(x+iy)) \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy$$

$$= \int_P \operatorname{tr}(\phi(r)^* N_{\pi,\alpha} \phi(r)) \Delta(r)^{-\alpha\nu/2}\, dr. \tag{4.2}$$

(Note that we do not assert the finiteness of the two integrals in (4.2); however, if one integral is infinite then so is the other.)

*Proof.* Let $\epsilon$ be any element of $P$. Then the function

$$f(r) = e^{-\operatorname{tr} \epsilon r} \pi(r^{1/2}) \phi(r), \qquad r \in P,$$

is integrable on $P$. Hence, its Laplace transform

$$F(z) = (2\pi)^{-nm/2} \int_P e^{i \operatorname{tr} zr} f(r)\, dr$$

is holomorphic on $H$ (cf., the proof of Theorem 9 in [2]). Since $F(z) = \Phi(z + i\epsilon)$ and $\epsilon$ is arbitrary in $P$, it follows that $\Phi$ is holomorphic on $H$.

By Theorem 3.6, the integral defining $\Gamma(2ir, \alpha, \pi)$ converges absolutely for all $r \in P$, and from (3.32) and Corollary 3.7

$$\Gamma(2ir, \alpha, \pi) = \Delta(r)^{-\alpha\nu/2} \pi(r^{-1/2}) N_{\pi,\alpha} \pi(r^{-1/2}) > 0 \tag{4.3}$$

for all $r \in P$. By (4.1)

$$\pi(y^{1/2}) \Phi(x+iy) = (2\pi)^{-nm/2} \int_P e^{i \operatorname{tr} xr} e^{-\operatorname{tr} yr} \pi(y^{1/2} r^{1/2}) \phi(r)\, dr.$$

Thus, by the Plancherel theorem for functions on the vector group $S$ (note that $\dim S = nm$),

$$\int_S \| \pi(y^{1/2}) \Phi(x+iy) \|^2\, dx = \int_P e^{-2 \operatorname{tr} yr} \| \pi(y^{1/2} r^{1/2}) \phi(r) \|^2\, dr \tag{4.4}$$

for all $y \in P$, although the integrals may be infinite. Since

$$\| \pi(y^{1/2} r^{1/2}) \phi(r) \|^2 = \operatorname{tr}(\phi(r)^* \pi(r^{1/2}) \pi(y) \pi(r^{1/2}) \phi(r))$$

it follows from Fubini's theorem that

$$\int_H \| \pi(y^{1/2})\, \Phi(x+iy)\|^2\, \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy$$
$$= \operatorname{tr} \left\{ \int_P \phi(r)^*\, \pi(r^{1/2}) \left( \int_P e^{-2\operatorname{tr} ry} \pi(y)\, \Delta(y)^{-m+\alpha\nu/2}\, dy \right) \pi(r^{1/2})\, \phi(r)\, dr \right\}.$$

Now

$$\int_P e^{-2\operatorname{tr} ry} \pi(y)\, \Delta(y)^{-m+\alpha\nu/2}\, dy = \Gamma(2ir, \alpha, \pi),$$

so by (4.3)

$$\int_H \| \pi(y^{1/2})\, \Phi(x+iy)\|^2\, \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy$$
$$= \int_P \operatorname{tr}(\phi(r)^*\, N_{\pi,\alpha} \phi(r))\, \Delta(r)^{-\alpha\nu/2}\, dr$$

which is equivalent to (4.2).

THEOREM 4.2. *Let $\pi$ be an* ifdp *representation of $A$ and $K_\pi$ the corresponding reduced Bessel function for $\mathbb{F}^{k\times n}$, let $\alpha$ be any real number such that $\omega(\pi) > (n-\alpha-1)\nu/2$, and set $\eta = (n-k-1+\alpha/2)\nu + 2$ and $\xi = 1 + (n-k-1+\alpha)\nu/2$. Then $N_{\pi,\alpha}$ is well defined and*

$$\int_P \operatorname{tr}(K_\pi(2r^{1/2})\, N_{\pi,\alpha} K_\pi(2r^{1/2})^*)\, \Delta(r)^{-\eta}\, dr$$
$$= c \left( \int_S \Delta(x^2+1)^{-\xi}\, dx \right) \left( \int_P e^{-2\operatorname{tr} y} \operatorname{tr}(\pi(y))\, \Delta(y)^{-\eta}\, dy \right) \tag{4.5}$$

*where $c = \beta^2(2\pi)^{-nm}$ with $\beta$ as in (2.22). The integrals in (4.5) are finite if and only if*

$$\tfrac{1}{2}(k\nu - 1) < \tfrac{1}{2}\alpha\nu < \omega(\pi) - 1 + \nu(k-n+1) \tag{4.6}$$

*in which case*

$$\int_P \| K_\pi(2r^{1/2})\|^2\, \Delta(r)^{-\eta}\, dr < \infty. \tag{4.7}$$

*Proof.* We apply Theorem 4.1 with $\phi(r) = (2\pi)^{nm/2}\, \Delta(r)^{-m+k\nu/2}\, K_\pi(2r^{1/2})^*$. By (2.22), (2.11), and (4.2)

$$\int_P \operatorname{tr}(K_\pi(2r^{1/2})\, N_{\pi,\alpha} K_\pi(2r^{1/2})^*)\, \Delta(r)^{-\eta}\, dr$$
$$= \beta^2(2\pi)^{-nm} \int_H |\, e^{-i\operatorname{tr} z^{-1}}\, |^2\, |\, \Delta(z^{-1})|^{k\nu} \operatorname{tr}(\pi(z^{-1*} y z^{-1}))\, \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy.$$

To evaluate the integral over $H$ we use three facts which are special cases of results in Section 7.

(4.8) The measure $\Delta(y)^{-2m}\,dx\,dy$ on $H$ is invariant under $z \to -z^{-1}$.

(4.9) $\mathrm{Im}(-z^{-1}) = z^{-1}(\mathrm{Im}\,z)z^{-1*}$ for $z \in H$.

(4.10) $(-z)^* \cdot (\mathrm{Im}(-z^{-1})) \cdot (-z) = \mathrm{Im}\,z$ for $z \in H$.

(Here, we have written $y$ as $\mathrm{Im}(z)$, where $z = x + iy$.) Making the substitution $z \to -z^{-1}$ and using (4.8)–(4.10), we find that the above integral over $H$ is

$$\begin{aligned}
&\int_H e^{-2\operatorname{tr} y} \mid \Delta(z)|^{k\nu} \operatorname{tr}(\pi(y))\, \Delta(z^{-1}yz^{-1*})^{m+\alpha\nu/2}\, \Delta(y)^{-2m}\, dx\, dy \\
&= \int_H e^{-2\operatorname{tr} y} \mid \Delta(x+iy)|^{k\nu-2m-\alpha\nu} \operatorname{tr}(\pi(y))\, \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy \\
&= \int_H e^{-2\operatorname{tr} y} \mid \Delta(y^{-1/2}xy^{-1/2}+i)|^{k\nu-2m-\alpha\nu} \operatorname{tr}(\pi(y))\, \Delta(y)^{k\nu-3m-\alpha\nu/2}\, dx\, dy \\
&= \left(\int_S \mid \Delta(x+i)|^{k\nu-2m-\alpha\nu}\, dx\right)\left(\int_P e^{-2\operatorname{tr} y} \operatorname{tr}(\pi(y))\, \Delta(y)^{k\nu-2m-\alpha\nu/2}\, dy\right) \\
&= \left(\int_S \Delta(x^2+1)^{-m+\nu(k-\alpha)/2}\, dx\right)\left(\int_P e^{-2\operatorname{tr} y} \operatorname{tr}(\pi(y))\, \Delta(y)^{-\eta}\, dy\right)
\end{aligned}$$

from which (4.5) is immediate.

Now, it follows from an application of "polar coordinates for $S$" (cf., [8] or [3; Exposé 10]) that $\int_S \Delta(x^2+1)^{-\sigma}\,dx < \infty$ for $\sigma > (1 + (n-1)\nu)/2$. Hence, the integral over $S$ on the right side of (4.5) is finite for $\alpha\nu > k\nu - 1$. The integral over $P$ on the right side of (4.5) can be written as $\operatorname{tr}(\Gamma(2i, 2k-n+1-\alpha-2/\nu, \pi))$ which converges for $\omega(\pi) > \nu(n-k-1+\alpha/2)+1$. Therefore, the right side of (4.5) is finite provided

$$\tfrac{1}{2}(k\nu - 1) < (\alpha\nu/2) < \omega(\pi) - 1 - \nu(n-k-1).$$

Notice that (4.6) implies that $\omega(\pi) > 2^{-1}(k\nu - 1) + 1 + \nu(n - k - 1 + 2^{-1}\alpha) = 2^{-1}\nu(n-\alpha-1) + \alpha\nu + 2^{-1}(n-k-2)\nu + 2^{-1} > 2^{-1}\nu(n-\alpha-1) + 2^{-1}\nu(n+k-2-\nu^{-1}) > (n-\alpha-1)\nu/2$, so (4.6) is consistent with the existence of $N_{\pi,\alpha}$.

Finally, for $\alpha$ and $\pi$ satisfying (4.6),

$$\int_P \| K(2r^{1/2})\, N_{\pi,\alpha}^{1/2} \|^2\, \Delta(r)^{-\eta}\, dr < \infty$$

which is equivalent to (4.7).

In the case $\alpha = k$, which is central to applications in Section 7, the square-integrability condition (4.7) on $K_\pi$ holds essentially for all $\pi$. For by the results of Examples 3.3–3.5, $\omega(\pi) \geqslant 0$ whenever $\pi$ is polynomial. Now, when $\alpha = k$, condition (4.6) takes the form

$$\omega(\pi) > 1 - \nu((\tfrac{1}{2}k) - n + 1), \tag{4.11}$$

and since $k \geqslant 2n$ this is a constraint on $\pi$ only for the single example $\mathbb{R}^{2n \times n}$. The specific result is as follows.

COROLLARY 4.3. *Let $\pi$ be an* ifdp *representation of $A$. When $\mathbb{F} = \mathbb{R}$ and $k = 2n$ assume in addition that $\omega(\pi) > 0$. Then*

$$\int_P \| K_\pi(2r^{1/2})\|^2 \, \Delta(r)^{-2m+k\nu/2} \, dr < \infty, \tag{4.12}$$

*or equivalently,*

$$\int_{A_0} \| K_\pi(2a)\|^2 \mid \Delta(a)|^{-2m+k\nu} \, d_*a < \infty. \tag{4.13}$$

Under the additional hypothesis that the restriction of $\pi$ to $\mathcal{O}_0$ is irreducible, formula (4.5) takes on a particularly simple form. We state the result for $\alpha = k$.

COROLLARY 4.4. *Let $\pi$ be as in Corollary 4.3, and assume that $\pi \mid \mathcal{O}_0$ is irreducible. Then the gamma function is scalar-valued, and*

$$\int_{A_0} \| K_\pi(2a)^* \, \xi \|^2 \mid \Delta(a)|^{-2m+k\nu} \, d_*a = c\gamma_n \left( \frac{\Gamma(2i, k - 2m/\nu, \pi)}{\Gamma(2i, k, \pi)} \right) \| \xi \|^2 \tag{4.14}$$

*for all $\xi \in \mathscr{L}$, and*

$$\int_{A_0} \| K_\pi(2a)^* \|^2 \mid \Delta(a)|^{-2m+k\nu} \, d_*a = c\gamma_n d_\pi^{-1} \left( \frac{\Gamma(2i, k - 2m/\nu, \pi)}{\Gamma(2i, k, \pi)} \right) \tag{4.15}$$

*where $d_\pi = \deg \pi$, $c = \beta_{k,n}^{-2} \beta_n (2\pi)^{-nm} \pi^{-kn\nu}$, and*

$$\gamma_n = \int_S \Delta(x^2 + 1)^{-m} \, dx.$$

We close this section with some remarks on the Laplace inversion of (2.22), which gives an alternate characterization of $K_\pi$.

COROLLARY 4.5. *Let $\pi$ be as in Corollary 4.3, and for $z \in H$ set $F(z) = e^{-i \operatorname{tr} z^{-1}} \Delta(iz^{-1})^{k\nu/2} \pi(z^{-1})$. Then for each $y \in P$ the function $x \to F(x + iy)$ is absolutely integrable on $S$, and*

$$K_\pi(2r^{1/2})^* = \beta_{k,n} (2\pi)^{-n(m+k\nu)/2} \, \Delta(r)^{m-k\nu/2} \, \pi(r^{-1/2}) \int_S e^{-i \operatorname{tr} zr} F(z) \, dx \tag{4.16}$$

*for all $r \in P$, where the integral is absolutely convergent and independent of $y$.*

*Proof.* Formal Fourier–Laplace inversion of formula (2.22) yields (4.16). To justify this use of the inversion formula it is sufficient to prove the integrability of $x \to F(x + iy)$. Since $|\exp(-i \operatorname{tr} z^{-1})| = |\exp(-\operatorname{tr} \operatorname{Im}(-z^{-1}))|$ it follows from (4.9) that it is a bounded function of $x$. Consequently, by the transformation $x \to y^{1/2} x y^{-1/2}$

$$\int_S \|F(x + iy)\| \, dx \leqslant C \int_S |\Delta(x + i)|^{-k\nu/2} \, \|\pi(x + i)^{-1}\| \, dx$$

$$\leqslant C \int_S \Delta(x^2 + 1)^{-k\nu/4} \, [\operatorname{tr} \pi(x^2 + 1)^{-1}]^{1/2} \, dx$$

where $C$ is a constant depending upon $y$. From results of Section 3, we can write $\pi(x) = (\det x)^s \, \pi_p(x)$ for $x \in S$, where $\pi_p$ is a polynomial representation of $A$ and $s \geqslant 0$. (In fact, if $\pi = \pi_\sigma$ for $\sigma \in D_p$, then $s = \omega(\pi)$ in the real and complex context and $s = s_{2n}$ in the quaternionic case.) Thus, since $\pi_p$ is polynomial, the mapping $x \to \pi_p(x^2 + 1)^{-1}$ is bounded on $S$, so

$$\int_S \|F(x + iy)\| \, dx \leqslant C \int_S \Delta(x^2 + 1)^{-((k\nu/4)+(s/2))} \, dx.$$

As in the proof of Theorem 4.2, this latter integral converges for $k\nu/4 + s/2 > [\nu(n - 1) + 1]/2$, or

$$s > 1 - ((k - 2n + 2)\nu/2). \tag{4.17}$$

Since $k \geqslant 2n$, (4.17) is satisfied for any $s \geqslant 0$, except when $\mathbb{F} = \mathbb{R}$ and $k = 2n$, in which case $s = 0$ is excluded.

## 5. The Hilbert Spaces $L_\Gamma^2$ and $\mathfrak{H}_{\pi,k}$

In this section we use a variant of the Laplace transform to construct certain Hilbert spaces of holomorphic functions on $H$. We then describe unitary operators on this space which are equivalent to the reduced Bessel transforms of Section 2.

Definition 5.1. Let $\pi$ be an ifdp representation of $A$, let $\mathscr{V}$ be any finite-dimensional complex Hilbert space, and let $\mathscr{L} = \mathscr{L}(\mathscr{V}, \mathscr{H}_\pi)$ be the space of linear transformations from $\mathscr{V}$ to $\mathscr{H}_\pi$. Define $L_\pi^2(A_0, \mathscr{V})$ to be the space of all Baire functions $f\colon A_0 \to \mathscr{L}$ such that

$$f(\tau a) = \pi(\tau) f(a) \tag{5.1}$$

for all $(\tau, a) \in \mathscr{O}_0 \times A_0$, and

$$\int_{A_0} \|f(a)\|^2 \, d_* a < \infty. \tag{5.2}$$

Modulo null functions, $L_\pi^2(A_0, \mathscr{V})$ is a Hilbert space which can be realized in many different ways, depending upon $k$, as a space of holomorphic functions on $H$.

THEOREM 5.2. *Let $k \geqslant 2n$ and $\pi$ any* ifdp *representation of $A$. Then for any $f \in L_\pi^2(A_0, \mathscr{V})$, $\mathscr{V}$ arbitrary,*

$$F(z) = \beta^{-1/2}(2\pi)^{-nm/2} \int_{A_0} e^{i \operatorname{tr} za'a} \mid \Delta(a)|^{k\nu/2} \pi(a') f(a) \, d_*a \tag{5.3}$$

*is absolutely convergent for all $z \in H$ and defines a holomorphic function $F: H \to \mathscr{L}$. Here, $\beta = \beta_n$ is the constant in* (2.12). *Furthermore,*

$$\int_H \operatorname{tr}(F(x+iy)^* \pi(y) F(x+iy)) \, \Delta(y)^{-2m+k\nu/2} \, dx \, dy$$

$$= \int_{A_0} \operatorname{tr}(f(a)^* \Gamma_{\pi,k} f(a)) \, d_*a < \infty \tag{5.4}$$

*whenever $\Gamma_{\pi,k} = N_{\pi,k-2m\nu^{-1}}$ exists. When $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, or when $k > 2n$, $\Gamma_{\pi,k}$ exists for all* ifdp *representations $\pi$. When $\mathbb{F} = \mathbb{R}$ and $k = 2n$, $\Gamma_{\pi,k}$ exists provided $\omega(\pi) > 0$.*

*Proof.* Let $\epsilon$ be an element of $P$ and set $H_\epsilon = \{z = x + iy \in H: y > \epsilon\}$. Then for all $z \in H_\epsilon$,

$$\int_{A_0} \| e^{i \operatorname{tr} za'a} \Delta(a)^{k\nu/2} \pi(a') f(a) \| \, d_*a$$

$$\leqslant \int_{A_0} e^{-\operatorname{tr} \epsilon a'a} \mid \Delta(a)|^{k\nu/2} \, \| \pi(a') \| \, \| f(a) \| \, d_*a$$

$$\leqslant \left( \int_{A_0} e^{-2 \operatorname{tr} \epsilon a'a} \mid \Delta(a)|^{k\nu} \, \| \pi(a') \|^2 \, d_*a \right)^{1/2} \left( \int_{A_0} \| f(a) \|^2 \, d_*a \right)^{1/2}.$$

Now, for any ifdp representation $\pi$

$$\int_{A_0} e^{-2 \operatorname{tr} \epsilon a'a} \mid \Delta(a)|^{k\nu} \, \| \pi(a') \|^2 \, d_*a$$

$$= \int_{A_0} e^{-2 \operatorname{tr} \epsilon a'a} \Delta(a'a)^{k\nu/2} \operatorname{tr}(\pi(a'a)) \, d_*a$$

$$= \beta \int_P e^{-2 \operatorname{tr} \epsilon r} \Delta(r)^{k\nu/2} \operatorname{tr}(\pi(r)) \, d_*r = \beta \operatorname{tr}(\Gamma(2i\epsilon, k, \pi)) < \infty,$$

so $F(z)$ exists and

$$\| F(z) \| \leqslant (2\pi)^{-nm/2} (\operatorname{tr} \Gamma(2i\epsilon, k, \pi))^{1/2} \, \| f \|_2 \tag{5.5}$$

for all $z \in H_\epsilon$. Since every point of $H$ is in some $H_\epsilon$, if follows that $F$ is a well-defined function on $H$. By (5.1) and (2.12),

$$F(z) = \beta^{1/2}(2\pi)^{-nm/2} \int_P e^{i\,\mathrm{tr}\,zr} \Delta(r)^{k\nu/4}\, \pi(r^{1/2})\, f(r^{1/2})\, d_*r \tag{5.6}$$

so from Theorem 4.1 with $\phi(r) = \beta^{1/2}\Delta(r)^{-m+k\nu/4} f(r^{1/2})$ we conclude that $F$ is holomorphic and

$$\int_H \mathrm{tr}(F(z)^*\, \pi(y)\, F(z))\, \Delta(y)^{-m+\alpha\nu/2}\, dx\, dy$$
$$= \beta \int_P \mathrm{tr}(f(r^{1/2})^*\, N_{\pi,\alpha}\, f(r^{1/2}))\, \Delta(r)^{-2m+(k-\alpha)\nu/2}\, dr \tag{5.7}$$

for $\omega(\pi) > (n - \alpha - 1)\nu/2$. Of course, in this general context the integrals in (5.7) may be infinite. However, if $(k - \alpha)\nu/2 = m$ then (5.7) is valid for $\omega(\pi) > 1 - (k - 2n + 2)\nu/2$. Since $k \geqslant 2n$, this is *a* constraint on the polynomial representation $\pi$ only when $\mathbb{F} = \mathbb{R}$ and $k = 2n$, in which case (5.7) is valid for $\omega(\pi) > 0$. Set $(k - \alpha)\nu/2 = m$. Then (5.7) becomes

$$\int_H \mathrm{tr}(F(z)^*\, \pi(y)\, F(z))\, \Delta(y)^{-2m+k\nu/2}\, dx\, dy$$
$$= \beta \int_P \mathrm{tr}(f(r^{1/2})^*\, N_{\pi,k-2m\nu^{-1}} f(r^{1/2}))\, \Delta(r)^{-m}\, dr \tag{5.8}$$

and by (5.1), (3.38), (2.12), and (2.11)

$$\int_H \mathrm{tr}(F(z)^*\, \pi(y)\, F(z))\, \Delta(y)^{-2m+k\nu/2}\, dx\, dy$$
$$= \int_{A_0} \mathrm{tr}(f(a)^*\, \Gamma_{\pi,k} f(a))\, d_*a$$
$$= \int_{A_0} \| \Gamma_{\pi,k}^{1/2} f(a)\|^2\, d_*a \leqslant \| \Gamma_{\pi,k}^{1/2} \|^2 \int_{A_0} \|f(a)\|^2\, d_*a < \infty.$$

This establishes (5.4) and completes the proof.

Notice from (3.29) and (3.37) that the gamma factor $\Gamma_{\pi,k}$ in (5.4) can be rewritten as

$$\Gamma_{\pi,k} = \Gamma(2i, -2m/\nu, \pi^{(k)}) = \int_P e^{-2\,\mathrm{tr}\,y} \pi^{(k)}(y)\, \Delta(y)^{-2m}\, dy \tag{5.9}$$

where

$$\pi^{(k)}(a) = \delta_k(a)\, \pi(a), \qquad a \in A \tag{5.10}$$

and $\delta_k$ is specified as follows: In the real case

$$\delta_k(a) = (\det a)^{k/2} \tag{5.11}$$

for $a \in GL(n, \mathbb{C})$. Note that in the real case $\pi^{(k)}$ is double-valued when $k$ is odd. In the complex case,

$$\delta_k(a) = (\det a_1)^k \tag{5.12}$$

for $a = (a_1, a_2) \in A$ with $a_1, a_2 \in GL(n, \mathbb{C})$. In the quaternionic case,

$$\delta_k(a) = (\det a)^k \tag{5.13}$$

for $a \in GL(2n, \mathbb{C})$.

Theorem 5.2 provides a unitary equivalence between two Hilbert spaces which should be made explicit. Recall that $L_\pi^2(A_0, \mathscr{V})$ has the usual inner product

$$(f_1 \mid f_2) = \int_{A_0} \operatorname{tr}(f_2(a)^* f_1(a))\, d_*a.$$

However, since $\| \Gamma_{\pi,k}^{-1/2} \|^{-1} \| f \|_2 \leqslant \| \Gamma_{\pi,k}^{1/2} f \|_2 \leqslant \| \Gamma_{\pi,k}^{1/2} \| \| f \|_2$, if follows that the inner product

$$(f_1 \mid f_2)_{\pi,k} = \int_{A_0} \operatorname{tr}(f_2(a)^* \Gamma_{\pi,k} f_1(a))\, d_*a \tag{5.14}$$

on $L_\pi^2(A_0, \mathscr{V})$ is equivalent to the usual one.

DEFINITION 5.3. Suppose $k \geqslant 2n$. Let $\pi$ be an ifdp representation of $A_0$. When $\mathbb{F} = \mathbb{R}$ and $k = 2n$ assume in addition that $\omega(\pi) > 0$. We denote by $L^2_{\pi,k,\mathscr{V}} = L_\pi^2(A_0, \mathscr{V}, \Gamma_{\pi,k})$ the space of all functions in $L_\pi^2(A_0, \mathscr{V})$ with inner product given by (5.14). Let $\mathfrak{H}_{\pi,k,\mathscr{V}}$ denote the space of all functions $F$ on $H$ of the form (5.3) with $f \in L^2_{\pi,k,\mathscr{V}}$, the inner product on $\mathfrak{H}_{\pi,k,\mathscr{V}}$ being

$$(F_1 \mid F_2)_{\pi,k} = \int_H \operatorname{tr}(F_2(x+iy)^* \pi^{(k)}(y) F_1(x+iy))\, \Delta(y)^{-2m}\, dxdy. \tag{5.15}$$

Finally, let $\mathscr{T}_{\pi,k,\mathscr{V}}$ be the mapping $f \to F$ given by (5.3).

For convenience of notation, we fix $\pi$, $k$, and $\mathscr{V}$; we define $\pi^{(k)}$ by (5.10); we set $\Gamma = \Gamma_{\pi,k}$ and let $L_\Gamma^2 = L^2_{\pi,k,\mathscr{V}}$, $\mathfrak{H} = \mathfrak{H}_{\pi,k,\mathscr{V}}$, and $\mathscr{T} = \mathscr{T}_{\pi,k,\mathscr{V}}$; we delete the subscripts in the inner products (5.14) and (5.15); and finally we define the measure $d_*z$ on $H$ by

$$d_*z = \Delta(y)^{-2m}\, dx\, dy. \tag{5.16}$$

COROLLARY 5.4. *$\mathscr{T}$ is a unitary map of $L_\Gamma^2$ onto $\mathfrak{H}$.*

Next we characterize $\mathfrak{H}$ as a reproducing kernel space. Such spaces are described quite generally in [9]. As applied to $\mathfrak{H}$ this means that the point evaluations

$$E_z: F \to F(z), \qquad z \in H, \tag{5.17}$$

are continuous linear operators from $\mathfrak{H}$ onto $\mathscr{L} = \mathscr{L}(\mathscr{V}, \mathscr{H}_\pi)$. The Hilbert space structure can then be characterized in terms of the kernel function

$$(z, w) \to E_z E_w{}^*.$$

In our case, however, one can make these ideas more concrete by starting from an interpretation of the defining relation (5.3) for $\mathfrak{H}$ as an inner product in $L_\Gamma{}^2$.

THEOREM 5.5. *Let $\pi$ be an* ifdp *representation of $A$. When $\mathbb{F} = \mathbb{R}$ and $k = 2n$ assume in addition that $\omega(\pi) > 0$.*

(i) *For each $(w, \xi) \in H \times \mathscr{L}$ the equation*

$$(f_{w,\xi})(a) = \beta^{-1/2}(2\pi)^{-nm/2}\, e^{-i \operatorname{tr} w^* a' a} \mid \Delta(a)|^{k\nu/2}\, \Gamma^{-1}\pi(a)\xi \tag{5.18}$$

*defines a function $f_{w,\xi} \in L_\Gamma{}^2$ such that for $F = \mathscr{T} f$*

$$(\xi \mid F(w)) = (f_{w,\xi} \mid f). \tag{5.19}$$

(ii) *For each $z \in H$, the integral*

$$Q_{\pi^{(k)}}(z) = (2\pi)^{-nm} \int_P e^{i \operatorname{tr} zr}\, \Gamma(2ir, -2m/\nu, \pi^{(k)})^{-1}\, dr \tag{5.20}$$

*converges absolutely and defines a holomorphic function $Q_{\pi^{(k)}}$ from $H$ to the space $\mathscr{L}_\pi$ of linear transformations on $\mathscr{H}_\pi$, in terms of which the function $F_{w,\xi} = \mathscr{T} f_{w,\xi}$ has the form*

$$F_{w,\xi}(z) = Q_{\pi^{(k)}}(z - w^*)\xi. \tag{5.21}$$

(iii) *Norm convergence in $\mathfrak{H}$ implies uniform pointwise convergence on compact subsets of $H$. In particular, the point evaluations $E_z$ are continuous linear operators from $\mathfrak{H}$ to $\mathscr{L}$, and*

$$E_z E_w{}^* \xi = Q_{\pi^{(k)}}(z - w^*)\xi \tag{5.22}$$

*for all $z, w \in H$ and $\xi \in \mathscr{L}$.*

(iv) *The subspace $\mathfrak{H}^0$ spanned by the functions $F_{w,\xi}$ for $(w, \xi) \in H \times \mathscr{L}$ is dense in $\mathfrak{H}$.*

(v) *Let $\mathscr{M}$ denote the Hilbert space of all Baire functions $F: H \to \mathscr{L}$ such that*

$$\int_H \operatorname{tr}(F(z)^* \pi^{(k)}(y) F(z))\, d_* z < \infty$$

*with inner product in $\mathscr{M}$ given by* (5.15). *Then the orthogonal projection $\mathscr{Q}$ of $\mathscr{M}$ onto the closed subspace $\mathfrak{H}$ is given by*

$$(\mathscr{Q}F)(z) = \int_H Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w)\, d_* w \tag{5.23}$$

*for all $F \in \mathscr{M}$. In particular, $Q_{\pi^{(k)}}$ is a reproducing kernel for the space $\mathfrak{H}$ in the sense that*

$$F(z) = \int_H Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w)\, d_* w \tag{5.24}$$

*for all $F \in \mathfrak{H}$.*

*Proof.* Let $(w, \xi) \in H \times \mathscr{L}$. By (3.38) it follows that $f_{w,\xi}$ satisfies (5.1), and by (2.12)

$$\begin{aligned} &\int_{A_0} \operatorname{tr}(f_{w,\xi}(a)^*\, \Gamma f_{w,\xi}(a))\, d_* a \\ &\quad = (2\pi)^{-nm} \int_P |\, e^{-i \operatorname{tr} w^* r}\, |^2\, \|\, \Gamma^{-1/2} \pi(r^{1/2}) \xi\, \|^2\, \Delta(r)^{k\nu/2}\, d_* r \\ &\quad \leqslant (2\pi)^{-nm}\, \|\, \Gamma^{-1/2}\, \|^2\, \|\, \xi\, \|^2 \operatorname{tr}(\Gamma(2ri \operatorname{Im} w, k, \pi)) \end{aligned}$$

which is finite by Theorem 3.6. Thus, $f_{w,\xi} \in L_\Gamma^2$. Formula (5.19) is immediate from (5.18) and (5.3).

Set $F_{w,\xi} = \mathscr{T} f_{w,\xi}$. Then $F_{w,\xi} \in \mathfrak{H}$ and by (5.3) and (5.18)

$$F_{w,\xi}(z) = \beta^{-1} (2\pi)^{-nm} \int_{A_0} e^{i \operatorname{tr}(z - w^*) a' a}\, |\, \Delta(a)|^{k\nu}\, \pi(a')\, \Gamma^{-1} \pi(a) \xi\, d_* a$$

the integral being absolutely convergent. By (3.32) and (5.9)

$$\pi(r^{-1/2})\, \Gamma \pi(r^{-1/2}) = \Delta(r)^{-m + k\nu/2}\, \Gamma(2ir, -2m/\nu, \pi^{(k)})$$

so by (2.12)

$$F_{w,\xi}(z) = (2\pi)^{-nm} \int_P e^{i \operatorname{tr}(z - w^*) r}\, \Gamma(2ir, -2m/\nu, \pi^{(k)})^{-1}\, \xi\, dr.$$

From the fact that this integral is absolutely convergent together with the fact that any element of $H$ can be written as $z - w^*$ for some $z, w$ in $H$, we see that (5.20) converges absolutely and that (5.21) holds. By Theorem 5.2, $Q_{\pi^{(k)}}$ is holomorphic.

We prove (iii). Let $f \in L_\Gamma^2$ and $F = \mathscr{T} f$, and suppose $\epsilon$ is in $P$. By (5.5) and (5.4)

$$\begin{aligned} F(z)\| &\leqslant (2\pi)^{-nm/2} \operatorname{tr}(\Gamma(2i\epsilon, k, \pi))^{1/2} \| f \|_2 \\ &\leqslant (2\pi)^{-nm/2} \operatorname{tr}(\Gamma(2i\epsilon, k, \pi))^{1/2} \| \Gamma^{-1/2} \| \| F \| \end{aligned}$$

for all $z \in H_\epsilon$. Thus, point evaluations $E_z$ are continuous; and since any compact subset of $H$ is contained in some $H_\epsilon$, it follows that norm convergence implies uniform convergence on compact sets.

To see that $E_z$ maps $\mathfrak{H}$ *onto* $\mathscr{L}$, suppose $\xi \in \mathscr{L}$ such that $(F(z) \mid \xi) = 0$ for all $F \in \mathfrak{H}$. Then

$$\begin{aligned} 0 = (F_{z,\xi}(z) \mid \xi) &= (Q_{\pi^{(k)}}(z - z^*)\xi \mid \xi) \\ &= (2\pi)^{-nm} \int_P e^{-2\operatorname{tr}(r \operatorname{Im} w)} \| \Gamma^{-1/2} \pi^{(k)}(r^{1/2})\xi \| \, d_* r, \end{aligned}$$

so $\Gamma^{-1/2}\pi^{(k)}(r^{1/2})\xi = 0$ for all $r$ and we conclude that $\xi = 0$.

To obtain (5.22) it is enough by (5.21) to prove that

$$E_w{}^*\xi = F_{w,\xi} \tag{5.25}$$

for all $\xi$. But for any $F \in \mathfrak{H}$, (5.19) implies $(E_w{}^*\xi \mid F) = (\xi \mid F(w)) = (f_{w,\xi} \mid f) = (F_{w,\xi} \mid F)$, which implies (5.25) and completes the proof of part (iii).

To see that $\mathfrak{H}^0$ is dense, it suffices by (5.25) to prove that there are no nonzero elements of $\mathfrak{H}$ orthogonal to $E_w{}^*\xi$ for all $w$, $\xi$. But this is clear from the equation $(F \mid E_w{}^*\xi) = (E_w F \mid \xi)$ and the fact that $E_w$ maps $\mathfrak{H}$ *onto* $\mathscr{L}$.

To prove (v), we first observe from (5.20) or (5.22) that $Q_{\pi^{(k)}}(z - w^*) = Q_{\pi^{(k)}}(w - z^*)^*$ for all $z, w \in H$. Then for any $F \in \mathfrak{H}$ and $\xi \in \mathscr{L}$

$$\begin{aligned} (F(z) \mid \xi) = (F \mid E_z{}^*\xi) &= (F \mid F_{z,\xi}) \\ &= \int_H \operatorname{tr}(\xi^* Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w))\, d_* w \\ &= \left( \int_H Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w)\, d_* w \mid \xi \right) \end{aligned}$$

and (5.24) follows. Finally, if $F \in \mathscr{M}$ is orthogonal to $\mathfrak{H}$, then

$$\begin{aligned} 0 = (E \mid E_z{}^*\xi) &= \int_H \operatorname{tr}(\xi^* Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w))\, d_* w \\ &= \left( \int_H Q_{\pi^{(k)}}(z - w^*)\, \pi^{(k)}(\operatorname{Im} w)\, F(w)\, d_* w \mid \xi \right) \end{aligned}$$

for all $\xi$ and $z$. It follows that (5.23) gives the orthogonal projection of $\mathscr{M}$ onto $\mathfrak{H}$. This completes the proof of the theorem.

COROLLARY 5.6. *The functions $f_{w,\xi}$ for $(w, \xi) \in H \times \mathscr{L}$ defined by* (5.18) *span a dense subspace of $L_\Gamma^2$.*

We remark in passing that $\mathfrak{H}$ may also be characterized by means of a Paley–Weiner type theorem as the space of all holomorphic functions in $\mathscr{M}$. However, we do not use this fact. For a proof along classical lines, see [7]. A simple representation theoretic proof can be obtained from results in the succeeding sections.

Next, we consider the special case of the preceding theory in which $\mathscr{V} = \mathscr{V}_\lambda$ as in Section 2. Then $\tilde{\mathscr{K}}_\pi^{-1}$, defined by (2.21), is a unitary operator on $L_\pi^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0^*)$. The following theorem gives a realization of $\tilde{\mathscr{K}}_\pi^{-1}$ on $L_\Gamma^2$, and in particular shows that $\tilde{\mathscr{K}}_\pi^{-1}$ remains unitary with respect to the inner product (5.14).

THEOREM 5.7. *Set $\mathscr{V} = \mathscr{V}_\lambda$, and let $\pi$ be as in Definition* 5.3. *Then the mapping $\mathscr{U}: F \to F^*$ defined by*

$$F^*(z) = \Delta(iz^{-1})^{k\nu/2}\, \pi(z^{-1})\, F(-z^{-1}) \tag{5.26}$$

*is a unitary operator on $\mathfrak{H}$, and the corresponding unitary operator $\mathscr{T}^{-1}\mathscr{U}\mathscr{T}$ on $L_\Gamma^2$ is $\tilde{\mathscr{K}}_\pi^{-1}$.*

*Proof.* Let $f \in L_\Gamma^2$ be such that $\Delta^{k\nu/2} f$ is integrable with respect to Haar measure on $A_0$, and let $F = \mathscr{T}f$. Then $\tilde{\mathscr{K}}_\pi^{-1} f \in L_\Gamma^2$ is given by (2.21). By (2.23), (5.3), and Fubini's theorem

$$\begin{aligned}
(\mathscr{T}\tilde{\mathscr{K}}_\pi^{-1} f)(z) &= \beta_n^{-1/2}(2\pi)^{-nm/2} \int_{A_0} e^{i\,\mathrm{tr}\, za'a} \mid \Delta(a)|^{k\nu/2}\, \pi(a')(\tilde{\mathscr{K}}_\pi^{-1} f)(a)\; d_*a \\
&= \beta_n^{-1/2}(2\pi)^{-nm/2} \int_{A_0} e^{i\,\mathrm{tr}\, za'a} \mid \Delta(a)|^{k\nu/2}\, \pi(a') \\
&\qquad \times \left(c_\pi \beta \int_{A_0} K_\pi(2ba')^* \mid \Delta(ab)|^{k\nu/2} f(b)\; d_*b\right) d_*a \\
&= \beta_n^{-1/2}(2\pi)^{-nm/2} \\
&\qquad \times \int_{A_0} \left(c_\pi \beta \int_{A_0} e^{i\,\mathrm{tr}\, za'a} \mid \Delta(a)|^{k\nu}\, \pi(a')\, K_\pi(2ba')^*\; d_*b\right) \\
&\qquad \times \mid \Delta(b)|^{k\nu/2} f(b)\; d_*b \\
&= \beta_n^{-1/2}(2\pi)^{-nm/2} \int_{A_0} e^{-i\,\mathrm{tr}\, z^{-1}b'b} \Delta(iz^{-1})^{k\nu/2} \\
&\qquad \times \pi(z^{-1}b') \mid \Delta(b)|^{k\nu/2} f(b)\; d_*b \\
&= \Delta(iz^{-1})^{k\nu/2}\, \pi(z^{-1})\, F(-z^{-1}) = F^*(z).
\end{aligned}$$

Thus, $F^{\#} \in \mathfrak{H}$, and by (4.8) and (4.10) with $z$ replaced by $-z^{-1}$

$$\begin{aligned} \| F^{\#} \|^2 &= \int_H \operatorname{tr}(F^{\#}(z)^* \,\pi(y)\, F^{\#}(z))\, \Delta(y)^{k\nu/2}\, d_* z \\ &= \int_H \operatorname{tr}(F(-z^{-1})^* \,\pi(z^{-1*} y z^{-1})\, F(-z^{-1}))\, \Delta(z^{-1*} y z^{-1})^{k\nu/2}\, d_* z \\ &= \int_H \operatorname{tr}(F(-z^{-1})^* \,\pi(\operatorname{Im}(-z^{-1}))\, F(-z^{-1}))\, \Delta(\operatorname{Im}(-z^{-1}))^{k\nu/2}\, d_* z \\ &= \int_H \operatorname{tr}(F(z)^* \,\pi(y)\, F(z))\, \Delta(y)^{k\nu/2}\, d_* z = \| F \|^2. \end{aligned}$$

Thus, $F \to F^{\#}$ is an isometry on a dense subspace of $\mathfrak{H}$, so it extends uniquely to a unitary operator $\mathscr{U}$ on $\mathfrak{H}$ with the property that $\mathscr{T}^{-1}\mathscr{U}\mathscr{T} = \tilde{\mathscr{K}}_\pi^{-1}$. Since norm convergence in $\mathfrak{H}$ implies pointwise convergence, it follows that $\mathscr{U}$ is given globally (that is, for any $F \in \mathfrak{H}$) by formula (5.26).

We conclude this section by computing the Bessel transform $\tilde{\mathscr{K}}_\pi$ explicitly on the dense subspace of $L_\Gamma{}^2$ of Corollary 5.6.

COROLLARY 5.7. *For $w \in H$ and $\xi \in \mathscr{L}_{\lambda,\pi}$,*

$$\tilde{\mathscr{K}}_\pi f_{w,\xi} = \Delta(-iw^{-1*})^{k\nu/2} f_{-w^{-1},\pi(w^{*-1})\xi}\,. \tag{5.27}$$

*Proof.* From (5.14) and (2.21)

$$((\tilde{\mathscr{K}}_\pi^{-1})^* f)(a) = \beta c_\pi \int_{A_0} \Gamma^{-1} K_\pi(2ab')\Gamma \mid \Delta(ab')|^{k\nu/2} f(b)\, d_* b. \tag{5.28}$$

We remark in passing that since $\tilde{\mathscr{K}}_\pi$ is unitary, (5.28) and (2.20) imply that

$$K_\pi(a)^* = \Gamma^{-1} K_\pi(-a')\Gamma \tag{5.29}$$

for all $a \in A_0$. Substitute $f_{w,\xi}$ for $f$ in (5.28). Then (5.28) becomes

$$(\tilde{\mathscr{K}}_\pi f_{w,\xi})(a) = \beta_{k,n}\beta_n^{-1/2} c_\pi \int_P \Gamma^{-1} K_\pi(2ar^{1/2}) \mid \Delta(ar)|^{k\nu/2}\, e^{-i\operatorname{tr} w^* r}\pi(r^{1/2})\xi\, d_* r$$

and (5.27) now follows from (6.25) of [1].

## 6. THE REPRESENTATIONS $R(\cdot, \pi, k)$ AND $T(\cdot, \pi, k)$ OF $G(n, \mathbb{F})$

This section deals with the infinite-dimensional representation theory of the group $G = G(n, \mathbb{F})$ of $2 \times 2$ matrices over $\mathbb{F}^{n\times n}$ such that

$$gpg' = p \tag{6.1}$$

where

$$p = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{6.2}$$

In particular, we construct representations $R(\cdot, \pi, k)$ and $T(\cdot, \pi, k)$ of $G$ which act in the spaces $L_\Gamma{}^2$ and $\mathfrak{H}_{\pi,k,\mathscr{V}_\lambda}$, respectively, and which are unitarily equivalent via the mapping $\mathscr{T}$. The Bessel transform $\mathscr{K}_\pi$ plays a decisive role in the definition and properties of $R(\cdot, \pi, k)$, and $T(\cdot, \pi, k)$ and will be used to calculate the kernel function. In order to avoid technical modifications involving cocycle representations, we initially assume in the real case that $k$ is even.

We begin by describing the action of $G$ on $H$. Let $M = (\mathbb{F}^{\mathbb{C}})^{n\times n}$, $z \to z'$ the complex-linear extension to $M$ of the involution in $\mathbb{F}^{n\times n}$, and let $z \to z^*$ be the conjugate-linear extension of the same involution. For $z = x + iy \in M$ with $x, y \in \mathbb{F}^{n\times n}$, we write $\bar{z} = x - iy$ and $\operatorname{Re} z = x$, $\operatorname{Im} z = y$. Note that $H = \{z \in M\colon z = z' \text{ and } \operatorname{Im} z > 0\}$. Since $z^* = \bar{z}'$, it follows that $\bar{z} = z^*$ for $z \in H$.

The matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in G \tag{6.3}$$

with $g_{ij} \in \mathbb{F}^{n\times n}$ defines a linear fractional transformation

$$z \circ g = (zg_{12} + g_{22})^{-1}(zg_{11} + g_{21}) \tag{6.4}$$

for all $z \in M$ such that $zg_{12} + g_{22} \in A$. In particular, if $g \in G$ and $z \in H$, then $zg_{12} + g_{22} \in A$ and $z \to z \circ g$ maps $H$ to $H$. Taking conjugates, we see that $z \to z \circ g$ also maps $H^*$ to $H^*$, $H^*$ being the lower half-plane of elements $z^*$ for $z \in H$. Moreover, $(z \circ g)^* = z^* \circ g$, and

$$\begin{aligned} z - w^* &= (zg_{12} + g_{22})(z \circ g - (w \circ g)^*)(wg_{12} + g_{22})^* \\ &= (w^*g_{12} + g_{22})(z \circ g - (w \circ g)^*)(z^*g_{12} + g_{22})^*. \end{aligned} \tag{6.5}$$

Finally, the mapping $m\colon H \times G \to A$ defined by

$$m(z, g) = zg_{12} + g_{22} \tag{6.6}$$

satisfies the multiplier identity

$$m(z, g_1g_2) = m(z, g_1)\, m(z \circ g_1, g_2) \tag{6.7}$$

for $z \in H$ and $g_1, g_2 \in G$. A reference for the above facts concerning the action of $G$ on $H$ is [2; Section 5]. There the arguments are carried out in detail over the complex field, but in point of fact they are valid for general $\mathbb{F}$.

LEMMA 6.1. *Let $\pi$ be an* ifdp *representation of $A$, let $\pi^{(k)}$ be given by* (5.10)–(5.13), *and let $\check{\pi}^{(k)}(a) = \pi^{(k)}(a'^{-1})$ for $a \in A$. Then the equation*

$$m_{\pi,k}(z, g) = \check{\pi}^{(k)}(m(z, g)) \tag{6.8}$$

*defines a continuous map $m_{\pi,k}$: $H \times G \to \mathscr{L}_\pi$, the space of linear transformations on $\mathscr{H}_\pi$, with the following properties: For each $g \in G$, $z \to m_{\pi,k}(z, g)$ is holomorphic on $H$, and*

$$m_{\pi,k}(z, g_1 g_2) = m_{\pi,k}(z, g_1)\, m_{\pi,k}(z \circ g_1, g_2) \tag{6.9}$$

*for all $z \in H$ and $g_1, g_2 \in G$. In addition,*

$$\pi^{(k)}(z - w^*)^{-1} = m_{\pi,k}(z, g)\, \pi^{(k)}(z \circ g - (w \circ g)^*)^{-1}\, m_{\pi,k}(w, g)^* \tag{6.10}$$

*for all $z, w \in H$ and $g \in G$.*

*Proof.* The identities (6.9) and (6.10) follow from (6.7) and (6.5), respectively, and the fact that $\check{\pi}^{(k)}$ is a representation. Since $\pi^{(k)}$ is holomorphic, $m_{\pi,k}$ is also holomorphic.

Now, let $\mathfrak{H} = \mathfrak{H}_{\pi,k,\mathscr{V}_\lambda}$, and when $\mathbb{F} = \mathbb{R}$ and $k = 2n$ assume that $\omega(\pi) > 0$. Then the equation

$$(T(g, \pi, k)F)(z) = m_{\pi,k}(z, g) F(z \circ g) \tag{6.11}$$

defines a unitary representation $T = T(\cdot, \pi, k)$ of $G$ on the space $\mathfrak{H}$. This can be proved directly from the characterization of $\mathfrak{H}$ as the space of all holomorphic functions in $\mathscr{M}$ (cf., the remark following Corollary 5.6). In the present context, however, it is appropriate to give a proof which involves $\tilde{\mathscr{H}}_\pi$ and additional structure in $G$.

A $2 \times 2$ matrix $g$ over $\mathbb{F}^{n\times n}$ lies in $G$ if and only if

$$g^{-1} = \begin{pmatrix} g'_{22} & -g'_{12} \\ -g'_{21} & g'_{11} \end{pmatrix}. \tag{6.12}$$

Hence, if $s = s' \in \mathbb{F}^{n\times n}$ it follows that

$$v(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \tag{6.13}$$

belongs to $G$. The set $V$ of all such matrices is a closed subgroup of $G$ isomorphic to the real vector group $S$ of self-adjoint elements of $\mathbb{F}^{n\times n}$. (Note that in Section 3 we used the symbol $V$ in a different context. Throughout the rest of the paper, $V$ refers to the group we have just defined.)

At this point, it is technically desirable to introduce a change in notation when $\mathbb{F} = \mathbb{R}$: *Throughout the rest of the paper, we reinterpret $A_0$ and $\mathscr{O}_0$ in the real case to be the full general linear group $GL(n, \mathbb{R})$ and the full orthogonal group*

$\mathcal{O}(n, \mathbb{R})$, *respectively*, rather than their identity components. A word of explanation is in order regarding the use of previous results in Sections 2–5, and Section 6 of [1], in the real case. These results all remain valid in the new notation. In fact, if $A_{00}$ and $\mathcal{O}_{00}$ denote the respective identity components of $A_0$ and $\mathcal{O}_0$, then $A_0 = A_{00} \cup \sigma A_{00}$ and $\mathcal{O}_0 = \mathcal{O}_{00} \cup \sigma\mathcal{O}_{00}$ where $\sigma = \operatorname{diag}(1,..., 1, -1)$. Then the mapping $f \to h = f|_{A_{00}}$, defined for functions $f$ on $A_0$ which are $\pi$-covariant with respect to $\mathcal{O}_0$ in the sense of (5.1), is a unitary operator from $L_\Gamma{}^2(A_0)$ to $L_\Gamma{}^2(A_{00})$, the inverse of which is

$$f(a) = \begin{cases} h(a) & \text{for} \quad a \in A_{00}, \\ \pi(\sigma)\, h(\sigma a) & \text{for} \quad a \in \sigma A_{00}. \end{cases}$$

Thus, for the sake of simplicity in avoiding the use of a multiplier (cf., Eq. (6.16) below) here and in succeeding sections, we prefer to work in the real case with $L_\Gamma{}^2(A_0)$ rather than $L_\Gamma{}^2(A_{00})$. In short, for general $\mathbb{F}$ we now have $A_0 = GL(n, \mathbb{F})$ and $\mathcal{O}_0 = \mathcal{O}(n, \mathbb{F})$.

Next, again by (6.12), if $a \in A_0$, the "diagonal" matrix

$$c(a) = \begin{pmatrix} a^\vee & 0 \\ 0 & a \end{pmatrix}, \qquad a^\vee = a'^{-1} \tag{6.14}$$

lies in $G$. The set of diagonal matrices in $G$ is a closed subgroup $C$, and $a \to c(a)$ $(a \in A_0)$ is an isomorphism of $A_0$ onto $C$. Moreover, $C$ normalizes $V$, and $L = VC$ is also a subgroup of $G$. In fact, an element $g$ of $G$ is in $L$ if and only if $g_{12} = 0$; so $L$ is the "lower triangular" (maximal parabolic) subgroup of $G$.

LEMMA 6.2. *The element $p$ defined by* (6.2) *also lies in $G$, and $LpV$ is a symmetric open set in $G$ whose complement is a set of Haar measure* 0. *The map $(l, v) \to lpv$ is a homeomorphism of $L \times V$ onto $LpV$, and*

$$G = (LpV)^2. \tag{6.15}$$

*Proof.* It is easy to check that $LpV = \{g \in G: g_{12} \in A_0\}$. Thus $LpV$ is open in $G$, and $LpV = (LpV)^{-1}$ by (6.12). It is an easy consequence of Bruhat's lemma [10] that $G - LpV$ is a set of measure 0. It follows from [2, Lemma 12] that $(l, v) \to lpv$ is a homeomorphism of $L \times V$ onto $LpV$. To prove (6.15), it is sufficient to show that

$$(LpV)g \cap (LpV) \neq \varnothing$$

for every $g$ in $G$. But this follows easily from the fact that the complements of the sets in question have measure 0.

For $\alpha \in \mathbb{C}$, let $\operatorname{sgn} \alpha = \alpha/|\alpha|$. Then from the fact that $L$ is the semidirect product of $V$ and $C$, it is easy to verify that the equation

$$(R(v(s)\, c(a), \pi, k)f)(b) = (\operatorname{sgn} \Delta(a))^{k\nu/2}\, e^{i \operatorname{tr} sb'b} f(ba) \tag{6.16}$$

defines a continuous representation of $L$ on $L_\Gamma{}^2$. In fact, $R = R(\cdot, \pi, k)$ is the product of a unitary character with a representation that is unitarily equivalent, in an obvious fashion, to a standard induced representation of $L$.

The surprising fact is that $R$ extends to a unitary representation of $G$.

THEOREM 6.3. *Let $\pi$ be an* ifdp *representation of $A$, and when $\mathbb{F} = \mathbb{R}$ and $k = 2n$ assume in addition that $\omega(\pi) > 0$. Then the representation $R(\cdot, \pi, k)$ of $L$ on $L_\Gamma{}^2$ extends to a continuous unitary representation, again denoted by $R(\cdot, \pi, k)$, of $G$ on $L_\Gamma{}^2$ with the property that for the element $p$ defined by* (6.2) *$R(p, \pi, k) = \Delta(i)^{-k\nu/2}\mathscr{K}_\pi^{-1}$. The corresponding unitary representation*

$$T(\cdot, \pi, k) = \mathscr{T}R(\cdot, \pi, k)\mathscr{T}^{-1}$$

*on $\mathfrak{H}$ is given by the equation*

$$(T(g, \pi, k)F)(z) = \pi^{(k)}(g'_{12}z + g'_{22})^{-1} F(z \circ g) \tag{6.17}$$

*for arbitrary $z$ in $H$ and $g$ in $G$.*

*Proof.* For the moment let $\mathfrak{F}$ denote the space of all holomorphic functions $F\colon H \to \mathscr{L}_{\lambda,\pi}$. If $F \in \mathfrak{F}$ and $g \in G$, it follows from Lemma 6.1 that the function

$$z \to m_{\pi,k}(z, g)\, F(z \circ g), \qquad z \in H$$

is also in $\mathfrak{F}$. Hence, for $g$ in $G$, we may define a linear transformation $S(g)$ on $\mathfrak{F}$ by setting

$$(S(g)F)(z) = m_{\pi,k}(z, g)\, F(z \circ g), \qquad z \in H.$$

It follows from (6.9) that

$$S(g_1 g_2) = S(g_1)\, S(g_2) \tag{6.18}$$

for all $g_1, g_2$ in $G$, and $S(1) = I$. Thus, $g \to S(g)$ $(g \in G)$ is a representation, in the purely algebraic sense, of $G$ on $\mathfrak{F}$.

To prove that (6.17) defines a unitary representation of $G$ on $\mathfrak{H}$, it suffices, in view of (6.18) and Lemma 6.2, to show that $\mathfrak{H}$ is invariant under the operators $S(g)$ and that these operators are unitary on $\mathfrak{H}$ for $g$ in $L \cup \{p\}$.

Now this is true for $S(p)$ by Theorem 5.7. To prove it for the operators $S(l)$, it suffices to show that $S(l)\mathscr{T} = \mathscr{T}R(l)$ for $l \in L$. But this is an easy consequence of the definition (6.16). For if $f \in L_\Gamma{}^2$ and $F = \mathscr{T}f$, then

$$\begin{aligned}(\mathscr{T}R(l)f)(z)& \\ &= \beta^{-1/2}(2\pi)^{-nm/2}\,(\operatorname{sgn}\Delta(a))^{k\nu/2}\int_{A_0} e^{i\operatorname{tr}(z+s)b'b}\,|\Delta(b)|^{k\nu/2}\,\pi(b')\,f(ba)\,d_*b\end{aligned}$$

$$= (\operatorname{sgn} \Delta(a))^{k\nu/2} \mid \Delta(a^{-1})|^{k\nu/2} \, \pi(a'^{-1}) \, \beta^{-1/2}(2\pi)^{-nm/2}$$

$$\times \int_{A_0} e^{i \operatorname{tr}(a^{-1}(z+s)a^{-1'}b'b)} \mid \Delta(b)|^{k\nu/2} \, \pi(b') \, f(b) \, d_* b$$

$$= (\operatorname{sgn} \Delta(a)^{k\nu/2} \mid \Delta(a^{-1})|^{k\nu/2} \, \pi(a'^{-1}) \, F(a^{-1}(z+s) \, a^{\vee})$$

$$= (\operatorname{sgn} \Delta(a^{\vee}))^{k\nu/2} \mid \Delta(a^{\vee})|^{k\nu/2} \, \pi(a^{\vee}) \, F(a^{-1}(z+s) \, a^{\vee})$$

$$= \pi^{(k)}(a^{\vee}) \, F(z \circ l) = (S(l) \mathscr{T} f)(z).$$

Therefore, (6.17) defines a unitary representation $T(\cdot, \pi, k)$ of $G$ on $\mathfrak{H}$ such that

$$T(l, \pi, k) = \mathscr{T} R(l, \pi, k) \mathscr{T}^{-1},$$

for $l \in L$, so $R(\cdot, \pi, k)$ is extended to a unitary representation of $G$ by the formula

$$R(g, \pi, k) = \mathscr{T}^{-1} T(g, \pi, k) \mathscr{T}$$

for arbitrary $g \in G$. Then $R(p, \pi, k) = \Delta(i)^{-k\nu/2} \tilde{\mathscr{K}}_{\pi}^{-1}$ by Theorem 5.7. To prove that $R = R(\cdot, \pi, k)$ is continuous, it suffices to prove continuity at $p$.

Suppose that $g_n \in G$ and $g_n \to p$. By Lemma 6.2, $LpV$ is an open neighborhood of $p$. Thus we may assume $g_n \in LpV$ for $n = 1, 2, \ldots$ . Then $g_n = l_n p v_n$ with $l_n \in L$ and $v_n \in V$, and for any $f_1, f_2$ in $L_{\Gamma}^2$

$$(R(g_n) f_1 \mid f_2) = (R(p) \, R(v_n) f_1 \mid R(l_n^{-1}) f_2).$$

Since $g_n \to p$ if follows from Lemma 6.2 that $l_n \to 1$ and $v_n \to 1$. Because $R$ is continuous on $L$, this implies $R(v_n) f_1 \to f_1$ and $R(l_n^{-1}) f_2 \to f_2$. Therefore, $(R(g_n) f_1 \mid f_2) \to (R(p) f_1 \mid f_2)$, which shows that $R(\cdot, \pi, k)$ is continuous and completes the proof.

We now turn to the computation of the function $\underset{\sim}{Q} = \underset{\sim}{Q}_{\pi(k)}$ which is given by (5.20).

THEOREM 6.4. *The function $\underset{\sim}{Q}$ defined in* (5.20) *and which determines the kernel function for* $\mathfrak{H}$, *as in* (5.22), *is given by the equation*

$$\underset{\sim}{Q}(z) = \gamma_{\pi} \pi^{(k)}(iz^{-1}), \qquad z \in H \tag{6.19}$$

*where*

$$\gamma_{\pi} = d_{\pi}^{-1} (2\pi)^{-nm} \operatorname{tr}(\Gamma(2i, k - 2m/\nu, \pi)^{-1} \, \Gamma(i, k\nu/2, \pi)) \tag{6.20}$$

*with* $d_{\pi} = \deg \pi$.

*Proof.* Fix $\pi$ and $k$, and for $z$ in $H$ and $g$ in $G$ let $M(z, g)$ denote the operator which maps $\xi$ in $\mathscr{L}_{\lambda,\pi}$ onto $m_{\pi,k}(z, g)\xi$. Then, by (6.17), the representation $T = T(\cdot, \pi, k)$ has the property that

$$E_z T(g) = M(z, g) E_{z \circ g}, \qquad z \in H, \quad g \in G$$

where $E_z$ is evaluation at $z$ on $\mathfrak{H}$ (cf. (5.17)). Therefore

$$T(g)^* E_z^* = E^*_{z\circ g} M(z, g)^*.$$

Replacing $z$ by $w$ in this equation and multiplying by the first, we find that

$$E_z E_w^* = M(z, g)\, E_{z\circ g} E^*_{w\circ g} M(w, g)^*.$$

Thus, by (5.22), the function $Q$, which is given by (5.20), has the property that

$$Q(z - w^*) = m_{\pi,k}(z, g)\, Q(z \circ g - (w \circ g)^*)\, m_{\pi,k}(w, g)^* \tag{6.21}$$

for all $z, w$ in $H$ and $g$ in $G$. For $g = c(a)$, as in (6.14), (6.21) asserts that

$$Q(z - w^*) = \pi^{(k)}(a^\vee)\, Q(a^{-1}(z - w^*)a^\vee)\, \pi^{(k)}(a^{-1}).$$

From this it follows that

$$Q(ciy) = \pi^{(k)}(y^{-1/2})\, Q(ci)\, \pi^{(k)}(y^{-1/2}) \tag{6.22}$$

for $y$ in $P$ and real $c > 0$.

Now let $K = \{k \in G\colon kk^* = 1\}$, the maximal compact subgroup of $G$. By (6.12), $K$ consists of all matrices

$$k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a, b \in \mathbb{F}^{n\times n}$ such that $aa' + bb' = 1$ and $ab' = ba'$. Thus, the mapping $k \to u = a + ib$ is an isomorphism of $K$ with the unitary subgroup $U_A = \{u \in A\colon uu^* = 1\}$ of $A$. $K$ may alternatively be viewed as the stability group of the point $i$ in $H$. For $i \circ k = (ib + a)^{-1}(ia - b) = u^{-1}(iu) = i$; and conversely if $g \in G$ is such that $i \circ g = i$, then (6.4) and (6.12) imply that $g \in K$. Hence, taking $z = w = i$ in (6.17) and $g = k \in K$, we see that

$$Q(2i) = m_{\pi,k}(i, k)\, Q(2i)\, m_{\pi,k}(i, k)^*.$$

By (6.8) and (6.6)

$$m_{\pi,k}(i, k) = \tilde{\pi}^{(k)}(u).$$

Thus, $Q(2i)$ commutes with the restriction of $\tilde{\pi}^{(k)}$ to the subgroup $U_A$ of $A$. Since $\tilde{\pi}^{(k)}$ is holomorphic and $U_A$ is a real form of $A$, the restriction of $\tilde{\pi}^{(k)}$ to $U_A$ remains irreducible, and we conclude that $Q(2i)$ is scalar. By (6.22), $Q(i)$ is also scalar and $Q(iy) = Q(i)\, \pi^{(k)}(y^{-1})$. It follows by analytic continuation in the "right half-plane" that $Q(z) = Q(i)\, \pi^{(k)}(iz^{-1})$.

We compute $Q(i)$. By (5.20) and (3.32)

$$Q(i) = (2\pi)^{-nm} \int_P e^{-\operatorname{tr} r} \Gamma(2ir, -2m/\nu, \pi^{(k)})^{-1} \, dr$$

$$= (2\pi)^{-nm} \int_P e^{-\operatorname{tr} r} \pi(r^{1/2}) \, \Gamma^{-1} \pi(r^{1/2}) \, \Delta(r)^{-m+k\nu/2} \, dr$$

where $\Gamma = \Gamma(2i, k - 2m/\nu, \pi)$. Since $Q(i)$ is scalar,

$$Q(i) = d_\pi^{-1} (2\pi)^{-nm} \int_P e^{-\operatorname{tr} r} \operatorname{tr}(\pi(r^{1/2}) \, \Gamma^{-1} \pi(r^{1/2})) \, \Delta(r)^{-m+k\nu/2} \, dr$$

$$= d_\pi^{-1} (2\pi)^{-nm} \operatorname{tr} \left( \Gamma^{-1} \int_P e^{-\operatorname{tr} r} \pi(r) \, \Delta(r)^{-m+k\nu/2} \, dr \right)$$

$$= d_\pi^{-1} (2\pi)^{-nm} \operatorname{tr}(\Gamma^{-1} \Gamma(i, k\nu/2, \pi)).$$

We close the section with some remarks on the preceding representation theory in the context of $\mathbb{F} = \mathbb{R}$ and $k$ odd. In that case, the presence of the principal branch of the square root in the multiplier (6.8) means that the multiplier identity (6.9) holds only up to a factor of $\pm 1$. Thus, (6.11) defines a cocycle representation $T = T(\cdot, \pi, k)$ of the real symplectic group $G = Sp(n, \mathbb{R})$, for which the cocycle has values $\pm 1$; that is to say,

$$T(g_1 g_2) = \epsilon(g_1, g_2) \, T(g_1) \, T(g_2) \tag{6.23}$$

for all $g_1, g_2 \in G$ where $\epsilon(g_1, g_2) = \pm 1$. Put another way, $T$ defines a unitary representation in the space $\mathfrak{H}$ of the *metaplectic group* $Mp(n, \mathbb{R})$, a central extension of $Sp(n, \mathbb{R})$ by the group $\{\pm 1\}$. Explicitly, the group law in $Mp(n, \mathbb{R})$ is

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \epsilon(g_1, g_2)) \tag{6.24}$$

for $g_1, g_2 \in G$ and $\epsilon_1, \epsilon_2 = \pm 1$, and the representation $T = T(\cdot, \pi, k)$ of $Mp(n, \mathbb{R})$ is given by

$$(T(g, \epsilon) F)(z) = \epsilon \tilde{\pi}^{(k)}(z g_{12} + g_{22}) \, F(z \circ g) \tag{6.25}$$

for $F \in \mathfrak{H}$.

## 7. Square-Integrability of $R(\cdot, \pi, k)$

Our purpose in this section is to prove that the representations $R(\cdot, \pi, k)$ are square-integrable and that square-integrability is a property that depends only upon the nature of the Bessel transform $\mathscr{K}_\pi$. Indeed, we show that the square-integrability of the representation is implied by the square-integrability of the Bessel function $K_\pi$ (Corollary 4.3).

We use two integration formulas. Let $ds$ be a Lebesgue measure on the vector group $S$ and $da$ (rather than $d_*a$, as previously used) a Haar measure on $A_0$. Since $L = VC$ is a semidirect product, the formula

$$\int_L f(l)\, dl = \int_S \int_{A_0} f(v(s)\, c(a))\, ds\, da \tag{7.1}$$

defines a right Haar measure on $L$, and (7.1) is valid for any nonnegative Baire function. Second, the formula

$$\int_G f(g)\, dg = \int_L \int_V f(l^{-1}pv)\, dl\, dv \tag{7.2}$$

in which $dl$ is defined by (7.1) and $dv = dv(s) = ds$, is valid with an appropriate normalization of Haar measure on $G$ for all nonnegative Baire functions. Although not standard, (7.2) follows readily from a well-known formula for $G$. (Simply notice that $L$ and $pVp^{-1}$ have a trivial intersection, and observe that Lemma 6.2 implies that $LpVp^{-1}$ is an open set in $G$ whose complement is a null set.)

The following theorem shows that $R = R(\cdot, \pi, k)$ is square-integrable in the sense that every matrix entry is in $L^2(G)$.

THEOREM 7.1. *Let $\pi$ be an* ifdp *representation of $A$, and in the case $k = 2n$ and $\mathbb{F}$ real assume that $\omega(\pi) > 0$. For any $f_1$ and $f_2$ in $L_\Gamma{}^2$*

$$\int_G |(R(g) f_1 \mid f_2)|^2\, dg \leqslant c \left(\int_{A_0} \| K_\pi(2a)\|^2\, \Delta(a)^{-2m+kv}\, da\right) \| f_1 \|^2\, \| f_2 \|^2 \tag{7.3}$$

*where $c = \beta_{k,n}^2 4^{nm} \pi^{n(2m-kv)} \| \Gamma^{1/2} \|^2 \| \Gamma^{-1/2} \|^2$. Whenever the representation $\tau \to \pi(\tau)$ of $\mathcal{O}_0$ is irreducible the precise formula*

$$\int_G |(R(g) f_1 \mid f_2)|^2\, dg = c_1 d_\pi^{-2} \left(\int_{A_0} \| K_\pi(2a)\|^2\, \Delta(a)^{-2m+kv}\, da\right) \| f_1 \|^2\, \| f_2 \|^2 \tag{7.4}$$

*holds for $f_1, f_2 \in L_\Gamma{}^2$, where $d_\pi = \deg \pi$ and $c_1 = \beta_{k,n}^2 4^{nm} \pi^{n(2m-kv)}$.*

In the proof we use the following lemma which, among other things illustrates the nonunimodular phenomenon of a primary representation, in fact, a finite multiple of an irreducible, possessing a dense subspace of square-integrable matrix entries without square-integrability for all matrix entries.

LEMMA 7.2. *For any $f_1$ and $f_2$ in $L_\Gamma{}^2$*

$$\int_L |(R(l) f_1 \mid f_2)|^2\, dl \leqslant (2\pi)^{nm}\, \beta_n \| f_1 \|^2 \int_{A_0} \| \Gamma^{1/2} f_2(a)\|^2\, | \Delta(a)|^{-2m}\, da. \tag{7.5}$$

*If $f_1$ and $f_2$ map $A_0$ into a d-dimensional subspace which is invariant and irreducible under the restriction of $\pi$ to $\mathcal{C}_0$, then*

$$\int_L |(R(l) f_1 \mid f_2)|^2 \, dl = (2\pi)^{nm} \beta_n d^{-1} \|f_1\|^2 \int_{A_0} \| \Gamma^{1/2} f_2(a)\|^2 \Delta(a)^{-2m} \, da. \quad (7.6)$$

*Proof.* Let $f_1$ and $f_2$ be in $L_\Gamma{}^2$. Then by (6.16), (5.14), (5.1), (3.38), and (2.12)

$$\begin{aligned}
&(R(v(s)\, c(a)) f_1 \mid f_2) \\
&\quad = (\operatorname{sgn} \Delta(a))^{k\nu/2} \int_{A_0} e^{i \operatorname{tr} sb'b} \operatorname{tr}(f_2(b)^* \, \Gamma f_1(ba)) \, db \\
&\quad = \beta(\operatorname{sgn} \Delta(a))^{k\nu/2} \int_P e^{i \operatorname{tr} sr} \operatorname{tr}(f_2(r^{1/2})^* \, \Gamma f_1(r^{1/2}a)) \, \Delta(r)^{-m} \, dr.
\end{aligned}$$

By (7.1), the Plancherel theorem for functions on $S$, (2.12), and the invariance of Haar measure

$$\begin{aligned}
\int_L |(R(l) f_1 \mid f_2)|^2 \, dl &= \int_{A_0} \int_S |(R(v(s)\, c(a)) f_1 \mid f_2)|^2 \, ds \, da \\
&= (2\pi)^{nm} \beta^2 \int_{A_0} \int_P | \operatorname{tr}(f_2(r^{1/2})^* \, \Gamma f_1(r^{1/2}a)|^2 \, \Delta(r)^{-2m} \, dr \, da \\
&= (2\pi)^{nm} \beta \int_{A_0} \int_{A_0} \operatorname{tr}(f_2(b)^* \, \Gamma f_1(ba))|^2 \, | \Delta(b)|^{-2m} \, db \, da \\
&= (2\pi)^{nm} \beta \int_{A_0} \int_{A_0} | \operatorname{tr}(f_2(b)^* \, \Gamma f_1(a))|^2 \, | \Delta(b)|^{-2m} \, db \, da.
\end{aligned}$$

By the Schwarz inequality

$$| \operatorname{tr}(f_2(b)^* \, \Gamma f_1(a))| \leqslant \| \Gamma^{1/2} f_1(a)\| \, \| \Gamma^{1/2} f_2(b)\|,$$

and

$$\int_L |(R(l) f_1 \mid f_2)|^2 \, dl \leqslant (2\pi)^{nm} \beta \int_{A_0} \| \Gamma^{1/2} f_1(a)\|^2 \, da \int_{A_0} \| \Gamma^{1/2} f_2(b)\|^2 \, | \Delta(b)|^{-2m} \, db$$

which proves (7.5).

Now suppose $\rho$ is an irreducible subrepresentation of $\pi \mid \mathcal{C}_0$ which acts in a $d$-dimensional subspace $Z$ of $\mathscr{H}_\pi$. In addition, assume $f_1$ and $f_2$ map $A_0$ into $Z$. Then by the above, (2.12), (3.32), and the Schur orthogonality relations

$$\begin{aligned}
&\int_L |(R(l)\,f_1 \mid f_2)|^2\,dl \\
&\quad = (2\pi)^{nm}\,\beta^2 \int_{A_0}\int_P \left(\int_{\mathcal{O}_0} |\operatorname{tr}(f_2(b)^*\,\Gamma^{1/2}\rho(\tau)\,\Gamma^{1/2}f_1(r^{1/2}))|^2\,d\tau\right) \\
&\qquad\quad \times \Delta(r)^{-m}\,dr \mid \Delta(b)|^{-2m}\,db \\
&\quad = (2\pi)^{nm}\,\beta^2 \int_{A_0}\int_P d^{-1}\,\|\,\Gamma^{1/2}f_1(r^{1/2})\|^2\,\|\,\Gamma^{1/2}f_2(b)\|^2\,\Delta(r)^{-m} \mid \Delta(b)|^{-2m}\,dr\,db \\
&\quad = (2\pi)^{nm}\,\beta d^{-1} \int_{A_0} \|\,\Gamma^{1/2}f_1(a)\|^2\,da \int_{A_0} \|\,\Gamma^{1/2}f_2(b)\|^2 \mid \Delta(b)|^{-2m}\,db
\end{aligned}$$

which establishes (7.6) and completes the proof of the lemma.

*Proof of Theorem* 7.1. Let $f_1$ and $f_2$ be any elements of $L_\Gamma{}^2$. Then by (7.2) and (7.5)

$$\begin{aligned}
\int_G |(R(g)\,f_1 \mid f_2)|^2\,dg &= \int_V\int_L |(R(l)\,f_2 \mid R(pv)\,f_1)|^2\,dl\,dv \\
&\leqslant (2\pi)^{nm}\,\beta_n\,\|\,f_2\,\|^2 \int_V\int_{A_0} \|\,\Gamma^{1/2}(R(pv)\,f_1)(a)\|^2 \mid \Delta(a)|^{-2m}\,da.
\end{aligned}$$

At this point we temporarily assume that $f_1$ lies in the dense subspace of $L_\Gamma{}^2$ for which $\mathscr{K}_\pi^{-1}$ is given by the integral (2.21). Then by Theorem 6.3, (6.16), (2.21), and (2.12)

$$(R(pv(s))\,f_1)(a) = \Delta(i)^{-k\nu/2}\,\beta c_\pi \int_P e^{i\,\operatorname{tr} st} K_\pi(2t^{1/2}a')^* \mid \Delta(at^{1/2})|^{k\nu/2} f_1(t^{1/2})\,\Delta(t)^{-m}\,dt$$

where $\beta = \beta_{k,n}$ and $c_\pi = \pi^{-kn\nu/2}$. Hence, by the Plancherel theorem for functions on $S$

$$\begin{aligned}
&\int_S \|\,\Gamma^{1/2}(R(pv(s))\,f_1)(a)\|^2\,ds \\
&\qquad = \beta^2(2\pi)^{nm}\,c_\pi{}^2 \int_P \|\,\Gamma^{1/2}K_\pi(2t^{1/2}a')^*\,f_1(t^{1/2})\|^2 \mid \Delta(a)|^{k\nu}\,\Delta(t)^{-2m+k\nu/2}\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_G |(R(g)\,f_1 \mid f_2)|^2\,dg \\
&\quad \leqslant c\beta_n\,\|\,f_2\,\|^2 \int_P\int_{A_0} \|\,\Gamma^{1/2}K_\pi(2t^{1/2}a')^*\,f_1(t^{1/2})\|^2 \mid \Delta(a)|^{-2m+k\nu}\,\Delta(t)^{-2m+k\nu/2}\,da\,dt \\
&\quad \leqslant c\beta_n\,\|\,f_2\,\|^2 \int_{A_0} \|\,\Gamma^{1/2}K_\pi(2a')^*\,\|^2 \mid \Delta(a)|^{-2m+k\nu}\,da \int_P \|\,f_1(t^{1/2})\|^2\,\Delta(t)^{-m}\,dt \\
&\quad \leqslant c\,\|\,\Gamma^{1/2}\,\|^2\,\|\,\Gamma^{-1/2}\,\|^2\,\|\,f_1\,\|^2\,\|\,f_2\,\|^2 \int_{A_0} \|\,K_\pi(2a')^*\,\|^2\,\Delta(a)^{-2m+k\nu}\,da
\end{aligned}$$

where $c = \beta_{k,n}^2 (2\pi)^{2nm} \pi^{-kn\nu}$. By (5.29), this proves (7.3) on a dense subspace of $L_\Gamma^2$, but it follows readily from Fatou's lemma that (7.3) holds for all $f_1, f_2 \in L_\Gamma^2$. Of course, the integral over $A_0$ on the right side of (7.3) is finite by Corollary 4.3.

Under the additional assumption that the restriction of $\pi$ to $\mathcal{O}_0$ is irreducible, we may replace the original estimate involving (7.5) by the exact value given in (7.6). Then for $f_1$ and $f_2$ in $L_\Gamma^2$ (with $f_1$ in the above-mentioned dense subspace)

$$\begin{aligned}
&\int_G |(R(g) f_1 \mid f_2)|^2 \, dg \\
&\quad = (2\pi)^{nm} \beta_n d_\pi^{-1} \|f_2\|^2 \int_V \int_{A_0} \| \Gamma^{1/2}(R(pv) f_1)(a)\|^2 \, |\Delta(a)|^{-2m} \, da \, dv \\
&\quad = (2\pi)^{nm} \beta_n d_\pi^{-1} \|f_2\|^2 \int_{A_0} \int_S \| \Gamma^{1/2}(R(pv(s)) f_1)(a)\|^2 \, |\Delta(a)|^{-2m} \, ds \, da \\
&\quad = (2\pi)^{nm} \beta_n d_\pi^{-1} \beta^2 (2\pi)^{nm} c_\pi^2 \|f_2\|^2 \int_{A_0} \int_P \| \Gamma^{1/2} K_\pi(2t^{1/2}a')^* f_1(t^{1/2})\|^2 \\
&\qquad \cdot |\Delta(a)|^{-2m+k\nu} \Delta(t)^{-2m+k\nu/2} \, dt \, da \\
&\quad = d_\pi^{-1} 4^{nm} \pi^{n(2m-k\nu)} \beta_{k,n}^2 \beta_n \|f_2\|^2 \int_P \int_{A_0} \| \Gamma^{1/2} K_\pi(2a')^* f_1(t^{1/2})\|^2 \\
&\qquad \cdot |\Delta(a)|^{-2m+k\nu} \, da \, d_* t.
\end{aligned}$$

Since $\tau \to \pi(\tau)$ ($\tau \in \mathcal{O}_0$) is irreducible, it follows that $\Gamma$ is a scalar matrix, and by (4.14), (4.15), and (5.29)

$$\begin{aligned}
&\int_{A_0} \| \Gamma^{1/2} K_\pi(2a')^* f_1(t^{1/2})\|^2 \, |\Delta(a)|^{k\nu-2m} \, da \\
&\qquad = d_\pi^{-1} \left( \int_{A_0} \| K_\pi(2a)\|^2 \, |\Delta(a)|^{k\nu-2m} \, da \right) \| \Gamma^{1/2} f_1(t^{1/2})\|^2.
\end{aligned}$$

Therefore, with $c_1$ as in the statement of the theorem,

$$\begin{aligned}
&\int_G |(R(g) f_1 \mid f_2)|^2 \, dg \\
&\quad = c_1 \beta_n d_\pi^{-2} \|f_2\|^2 \int_{A_0} \| K_\pi(2a)\|^2 \, |\Delta(a)|^{k\nu-2m} \, da \int_P \| \Gamma^{1/2} f_1(t^{1/2})\|^2 \, d_* t \\
&\quad = c_1 d_\pi^{-2} \|f_1\|^2 \|f_2\|^2 \int_{A_0} \| K_\pi(2a)\|^2 \, |\Delta(a)|^{k\nu-2m} \, da.
\end{aligned}$$

As before, we obtain (7.4) for all $f_1 \in L_\Gamma^2$, and this completes the proof.

We conclude with some additional remarks. The space $L_\Gamma{}^2$ consists of functions $f\colon A_0 \to \mathscr{L}_{\lambda,\pi}$ where $\mathscr{L}_{\lambda,\pi} = \mathscr{L}(\mathscr{V}_\lambda^{\frown}, \mathscr{H}_\pi)$. Let $d_\lambda = \deg \lambda$. Then $L_\Gamma{}^2$ is isomorphic to the direct sum of $d_\lambda$ copies of the space $L^2_{\Gamma,1}$ of functions $f\colon A_0 \to \mathscr{H}_\pi$ in which the covariance condition and inner product are given by (5.1) and (5.14), respectively. Consequently, the representation $R(\cdot, \pi, k)$ is equivalent to the direct sum of $d_\lambda$ copies of the analogous representation $R^1(\cdot, \pi, k)$ acting in the space $L^2_{\Gamma,1}$. Similarly, $T(\cdot, \pi, k)$, which acts in a space $\mathfrak{H}$ of functions with values in $\mathscr{L}_{\lambda,\pi}$, is equivalent to the direct sum of $d_\lambda$ copies of the analogous representation (given by formula (6.11)) $T^1(\cdot, \pi, k)$ which acts in a space $\mathfrak{H}_1$ of $\mathscr{H}_\pi$-valued holomorphic functions on $H$. The representations $T^1(\cdot, \pi, k)$ are easily seen to constitute a portion of the holomorphic discrete series for $G$. Hence, they and the equivalent representations $R^1(\cdot, \pi, k)$ are irreducible. Using this, one may of course improve the estimate given in (7.3). In the sequel, we shall use the notation $R(\cdot, \pi^{(k)})$ and $T(\cdot, \pi^{(k)})$ in place of $R^1(\cdot, \pi, k)$ and $T^1(\cdot, \pi, k)$, respectively. This is consistent with the notation in Section 1 (viz., (1.11)–(1.13)) and Section 8 (Theorem 8.3 *et seq.*).

Finally, we can observe from the form of the representations $R(\cdot, \pi^{(k)})$ that the restriction to the maximal parabolic subgroup $L$ is in general not irreducible. Indeed, it is not hard to see that the restriction of $R(\cdot, \pi^{(k)})$ to $L$ decomposes as a discrete direct sum

$$R(l; \pi^{(k)}) \cong n_1 R_1(l) \oplus n_2 R_2(l) \oplus \cdots \oplus n_q R_q(l) \tag{7.7}$$

where the representations $R_j$ $(1 \leqslant j \leqslant q)$ of $L$ are irreducible. In fact, the decomposition (7.7) mirrors the primary decomposition

$$\rho = n_1 \rho_1 \oplus \cdots \oplus n_q \rho_q \tag{7.8}$$

of the restriction $\rho$ of $\pi$ to the subgroup $\mathcal{O}_0$. This is a special case of results in [7].

## 8. Holomorphic Discrete Series for $G$ and Decomposition of Metaplectic Representations

In this section we review the construction of certain metaplectic representations of $G = G(n, \mathbb{F})$, analogous to those first considered by Shale [11], Segal [12], and Weil [13], and show how such representations decompose in terms of the representations $R(\cdot, \pi, k)$ described in Section 6.

We begin with a brief outline of the construction of a collection of metaplectic representations $\mathscr{R}_\alpha$ which act in the space $L^2(X)$, $X = \mathbb{F}^{k\times n}$. Since this construction is by now well known, we shall omit the details. For the complex field a more systematic development can be found in [2, 14], and in the general context in Weil's paper [13]. Although much of what we do initially is valid more generally, we shall assume, as we have throughout, that $k \geqslant 2n$. We use the notation of Section 1.

Let $Y = X \times X$. Then $Z = Y \times \mathbb{R}$, equipped with the law of composition

$$(y_1, t_1)(y_2, t_2) = (y_1 + y_2, t_1 + t_2 - \operatorname{Re} \operatorname{tr}(y_1 p y_2')) \tag{8.1}$$

where $p$ is given by (6.2), is a nilpotent (Heisenberg) group. Its center is $\{0\} \times \mathbb{R}$. Then each $g \in G$ determines an automorphism

$$\tilde{g}: (y, t) \to (yg^{-1}, t) \tag{8.2}$$

of $Z$ which leaves the center pointwise fixed.

Now, to each nonzero real number $\alpha$, there corresponds an irreducible unitary representation $\mathscr{S}_\alpha$ of $Z$ which acts on the space $L^2(X)$ by the formula

$$(\mathscr{S}_\alpha(x_1, x_2; t)f)(x) = e^{i\alpha\{t + \operatorname{Re}\operatorname{tr}[(2x+x_2)'x_1]\}} f(x + x_2) \tag{8.3}$$

for $x_1, x_2 \in X$, $t \in \mathbb{R}$. These representations $\mathscr{S}_\alpha$ comprise the Plancherel dual of $Z$. That is to say, any irreducible infinite-dimensional unitary representation of $Z$ is unitarily equivalent to precisely one of the representations $\mathscr{S}_\alpha$; specifically, to that $\mathscr{S}_\alpha$ with which it agrees on the center. As an immediate consequence there exists one cocycle representation of $G$ for each $\alpha$.

PROPOSITION 8.1. *Let $\alpha \neq 0$ be a real number. For each $g \in G$ there exists a unitary operator $\mathscr{R}_\alpha(g)$ on $L^2(X)$, unique up to scalar factors of absolute value one, such that*

$$\mathscr{R}_\alpha(g)\, \mathscr{S}_\alpha(z)\, \mathscr{R}_\alpha(g)^{-1} = \mathscr{S}_\alpha(\tilde{g}z) \tag{8.4}$$

*for all $z \in Z$. Moreover, given particular choices for $\mathscr{R}_\alpha(g_1)$ and $\mathscr{R}_\alpha(g_2)$ there exists a scalar $c(g_1, g_2)$ such that*

$$\mathscr{R}_\alpha(g_1 g_2) = c(g_1, g_2)\, \mathscr{R}_\alpha(g_1)\, \mathscr{R}_\alpha(g_2). \tag{8.5}$$

Thus, by (8.5) $\mathscr{R}_\alpha$ defines a cocycle representation of $G$ with cocycle $c$. In the complex or quaternionic cases, or the real case with $k$ even, the operators $\mathscr{R}_\alpha(g)$ can be specified once and for all in such a way that $\mathscr{R}_\alpha$ becomes a representation in the usual sense.

THEOREM 8.2. *For $l = v(s)\, c(a) \in L$ let*

$$(\mathscr{R}_\alpha(l)f)(x) = \Delta(a)^{k\nu/2}\, e^{i\alpha \operatorname{tr}(sx'x)} f(xa) \tag{8.6}$$

*and for $p$ given by* (6.2) *let*

$$(\mathscr{R}_\alpha(p)f)(x) = \Delta(i\alpha)^{-k\nu/2} \hat{f}(-\alpha x) \tag{8.7}$$

*with $f$ as in* (2.4). *Then the operators on* $L^2(X)$ *defined by* (8.6) *and* (8.7) *satisfy* (8.4), *and the equation*

$$\mathscr{R}_\alpha(lpv) = \mathscr{R}_\alpha(l)\,\mathscr{R}_\alpha(p)\,\mathscr{R}_\alpha(v) \tag{8.8}$$

*determines* $\mathscr{R}_\alpha$ *on* $LpV$. *Furthermore, in the complex and quaternionic cases, or in the real case when $k$ is even,* $\mathscr{R}_\alpha$ *extends uniquely from* $LpV$ *to a unitary representation of* $G$. *When* $\mathbb{F} = \mathbb{R}$ *and $k$ is odd,* $\mathscr{R}_\alpha$ *extends uniquely to a representation of the metaplectic group* $Mp(n, \mathbb{R})$.

For an outline of the proof see [2]. For the actual calculations we refer to [13] (cf., also [14]). As regards the real case with $k$ odd, it is readily seen from (8.6) that the square root creates an obstruction to "trivializing" the cocycle $c$ (cf., the end of Section 6).

In what follows we shall only treat the case in which $\alpha = 1$. This is no essential loss of generality for it is easy to verify from (8.6) and (8.7) that $\mathscr{R}_{\alpha_1}$ and $\mathscr{R}_{\alpha_2}$ are equivalent if and only if $\alpha_1$ and $\alpha_2$ have the same sign. Moreover, $\mathscr{R}_{\pm 1}$ are related by an automorphism of $G$; namely $\mathscr{R}_{-1}(g) = \mathscr{R}_1(\sigma g^\vee \sigma)$ where

$$\sigma = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

Thus, in what follows we set $\mathscr{R} = \mathscr{R}_1$ and we refer to $\mathscr{R}$ as the *metaplectic representation of* $G$ (even though it is double-valued when $X = \mathbb{R}^{k\times n}$ with $k$ odd). When needed for emphasis we shall write $\mathscr{R} = \mathscr{R}^{(k)}$ to indicate the dependence upon $k$.

We now obtain the primary decomposition of $\mathscr{R}$. To do so, one need only combine the representation theory of Section 6 of this paper with the harmonic analysis of Section 6 of [1]. In fact, in view of formula (8.7) the decomposition of $\mathscr{R}$ includes as a special case the Bessel function results in Section 1 (cf., (2.5) and (2.16)). As is seen from Theorem 6.1 of [1] and Theorem 6.3 of this paper, the case in which $X = \mathbb{R}^{2n\times n}$ is somewhat exceptional. Hence, so as not to obscure the main ideas, we initially exclude the case in which $\mathbb{F} = \mathbb{R}$ and $k = 2n$. That example will be discussed at the very end. Recall that the representation $R(\cdot, \pi, k)$ of $G$ on the space $L_\Gamma{}^2$ is defined by (6.16) and Theorem 6.3.

THEOREM 8.3. *Let* $\mathscr{R}^{(k)}$ *be the metaplectic representation of* $G$ *on the space* $L^2(\mathbb{F}^{k\times n})$ *with* $k \geqslant 2n$, *and suppose in the real case that* $k > 2n$. *Then*

$$\mathscr{R}^{(k)} \cong \sum_\pi \oplus\, R(\cdot, \pi, k) \tag{8.9}$$

*where the sum extends over all* ifdp *representations of* $A$. *Moreover,*

$$R(\cdot, \pi, k) \cong d_\lambda R(\cdot, \pi^{(k)}), \qquad d_\lambda = \deg \lambda(\cdot, \pi), \tag{8.10}$$

*where $R(\cdot, \pi^{(k)})$ is an irreducible square-integrable representation of $G$ which is unitarily equivalent to a representation in the holomorphic discrete series.*

In short, $\mathscr{R}$ decomposes discretely into representations in the holomorphic discrete series, each irreducible constituent appearing with multiplicity equal to the degree of the representation $\lambda = \lambda(\cdot, \pi)$ of $U$. Of course, in the real case with $k$ odd we are actually dealing with holomorphic discrete series of a two-fold covering of $G = Sp(n, \mathbb{R})$.

*Proof.* From formulas (8.6) and (8.7) it is easily seen that $\mathscr{R}$ commutes with the left regular representation of $U$ on $L^2(X)$. Hence, by [1; Section 1]

$$\mathscr{R} = \sum_{\gamma \in \hat{U}_X} \oplus \mathscr{R}(\cdot, \lambda) \tag{8.11}$$

where $\mathscr{R}(\cdot, \lambda)$ acts by formulas (8.6) and (8.7) on the space $L_\lambda{}^2(X, \mathscr{L}_\lambda)$ (cf., Definition 2.1). By [1, Theorem 6.1] $\pi \to \lambda = \lambda(\cdot, \pi)$ is a bijection from the ifdp representations $\pi$ of $A$ onto $\hat{U}_X$; and by [1; Corollary 6.5], (8.6), and (8.7), $\mathscr{R}(\cdot, \lambda)$ is unitarily equivalent to a unitary representation $\tilde{R}(\cdot, \pi)$ which acts in the space $L_\pi{}^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0{}^*)$ of Section 2 by the formulas

$$(\tilde{R}(l, \pi)f)(b) = (\operatorname{sgn} \Delta(a))^{k\nu/2}\, e^{i \operatorname{tr}(sb'b)} f(ba) \tag{8.12}$$

for $l = v(s)\, c(a) \in L$, and

$$(\tilde{R}(p, \pi)f)(b) = \Delta(i)^{-k\nu/2}\, (\tilde{\mathscr{K}}_\pi^{-1} f)(b). \tag{8.13}$$

Notice that by (6.16) and Theorem 6.3 the unitary representations $\tilde{R}(\cdot, \pi)$ and $R(\cdot, \pi, k)$ of $G$ on the spaces $L_\pi{}^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0{}^*)$ and $L_\Gamma{}^2$, respectively, are given by the same formulas, so the identity mapping (which is not necessarily unitary, but is bounded and has a bounded inverse, see Definition 5.3) is a bounded invertible intertwining operator. Thus, the unitary part of the polar decomposition of the identity map from $L_\pi{}^2(A_0, \mathscr{L}_{\lambda,\pi}, T_0{}^*)$ to $L_\Gamma{}^2$ yields the unitary equivalence of $\tilde{R}(\cdot, \pi)$ with $R(\cdot, \pi, k)$. By Theorem 7.1, $R(\cdot, \pi, k)$ is square-integrable, and the remainder of the proof follows from the remark on holomorphic discrete series at the end of Section 7.

We next comment upon how much of the holomorphic discrete series of $G$ appears in the primary decomposition (8.9) of $\mathscr{R}^{(k)}$.

First, observe that the proof of Theorem 5.2 is valid even when $\pi$ is not polynomial. In fact, a slight modification in formula (5.3) leads to the following result.

THEOREM 8.4. *Suppose $\pi$ is any* ihfd *representation of $A$ such that $\omega(\pi) >$*

*$(n-1)\nu/2$, and let $L_\pi^2(A_0)$ be the space of all square-integrable Baire functions $f\colon A_0 \to \mathscr{H}_\pi$ satisfying* (5.1). *Then for $f \in L_\pi^2(A_0)$*

$$F(z) = \beta^{-1/2}(2\pi)^{-nm/2} \int_{A_0} e^{i\,\operatorname{tr} za'a}\pi(a')\, f(a)\, d_*a \tag{8.14}$$

*defines a holomorphic function $F\colon H \to \mathscr{H}_\pi$. Furthermore, if $\omega(\pi) > m + (n-1)\nu/2$ with $m$ as in* (2.12), *then*

$$\int_H \operatorname{tr}(F(z)^*\, \pi(y)\, F(z))\, d_*z = \int_{A_0} \operatorname{tr}(f(a)^*\, N_{\pi,-2m/\nu} f(a))\, d_*a \tag{8.15}$$

*and the integrals are finite.*

Definition 8.5. Let $\pi$ be an ihfd representation of $A$ such that $\omega(\pi) > (n-1)\nu/2$, let $\mathscr{T} = \mathscr{T}_\pi$ be the mapping $f \to F$ given by (8.14), and let $\mathfrak{H}_\pi = \mathscr{T}(L_\pi^2(A_0))$. $\mathfrak{H}_\pi$ is a nonzero complex vector space of holomorphic functions on $H$ with values in $\mathscr{H}_\pi$. When $\omega(\pi) > m + (n-1)\nu/2$, $\mathfrak{H}_\pi$ is given the inner product

$$(F_1 \mid F_2)_\pi = \int_H \operatorname{tr}(F_2(z)^*\, \pi(y)\, F_1(z))\, d_*z. \tag{8.16}$$

Note that $m + (n-1)\nu/2 = (n-1)\nu + 1$.

When $\omega(\pi) > m + (n-1)\nu/2$, the results of Rossi and Vergne [7] show that $\mathfrak{H}_\pi$ can be characterized as the space of all holomorphic functions $F\colon H \to \mathscr{H}_\pi$ for which the left side of (8.15) is finite. This leads easily to the construction of representations $T(\cdot, \pi)$ in $\mathfrak{H}_\pi$ given by the formula

$$(T(g, \pi)F)(z) = \pi(g'_{12}z + g'_{22})^{-1} F(z \circ g). \tag{8.17}$$

The representations $T(\cdot, \pi)$ comprise the full holomorphic discrete series for $G$. In particular, the full holomorphic discrete series is parametrized by the ihfd representations $\pi$ of $A$ such that

$$\omega(\pi) > (n-1)\nu + 1. \tag{8.18}$$

Of course, the representations $R(\cdot, \pi) = \mathscr{T}_\pi T(\cdot, \pi)\mathscr{T}_\pi^{-1}$ give an equivalent realization of the holomorphic discrete series.

By comparing formulas (8.17) and (6.17), we can easily determine which of the representations in the holomorphic discrete series are related to the metaplectic representation $\mathscr{R}^{(k)}$; or equivalently to the Fourier transform on $\mathbb{F}^{k\times n}$. The result is as follows.

Corollary 8.6. *Let $\pi_0$ be an* ihfd *representation of $A$. Then the representation $T(\cdot, \pi_0)$ in the holomorphic discrete series is equivalent to a subrepresentation* *of $\mathscr{R}^{(k)}$ if and only if $\pi_0 = \pi^{(k)}$ for some polynomial representation $\pi$ such that*

$$\omega(\pi^{(k)}) > (n-1)\nu + 1 \tag{8.19}$$

*In fact, with notation as in* (8.9) *and* (8.10), $R(\cdot, \pi^{(k)}) \cong T(\cdot, \pi^{(k)})$.

Note that since $\omega(\pi^{(k)}) = \omega(\pi) + k\nu/2$, (8.19) implies $\omega(\pi^{(k)}) \geqslant k\nu/2$. In the real, complex, and quaternionic cases individually, Corollary 8.6 can be rephrased more explicitly in terms of highest weights. We use the notation of Examples 3.3–3.5.

EXAMPLE 8.7. Let $\mathbb{F} = \mathbb{R}$. Then $G = Sp(n, \mathbb{R})$ for $k$ even, and $G = Mp(n, \mathbb{R})$ for $k$ odd. Let $\pi$ be an ifdp representation of $A$. Then by (5.11) and Example 3.3, $\omega(\pi^{(k)}) = \omega(\pi) + k/2$ and $\omega(\pi) \geqslant 0$. Now, by (8.19), $\pi_0 = \pi^{(k)}$ indexes a member of the holomorphic discrete series if and only if

$$\omega(\pi) + k/2 > n. \tag{8.20}$$

Thus, all representations in the holomorphic discrete series appear in $\mathscr{R}^{(k)}$ except those indexed by highest weights such that

$$n < \omega(\pi) < k/2. \tag{8.21}$$

EXAMPLE 8.8. Let $\mathbb{F} = \mathbb{C}$. Then $G = U(n, n)$. Let $\pi$ be an ifdp representation of $A$ with highest weight $\sigma = (s_1, \ldots, s_n; t_1, \ldots, t_n)$. Then by (5.12) and Example 3.4, $\omega(\pi^{(k)}) = s_n + k + t_n$, $s_n \geqslant 0$ and $t_n \geqslant 0$. By (8.19), $\pi_0 = \pi^{(k)}$ indexes a member of the holomorphic discrete series if and only if

$$s_n + k + t_n > 2n - 1. \tag{8.22}$$

Thus, a member $T(\cdot, \pi_0)$ of the holomorphic discrete series occurs in the reduction of $\mathscr{R}^{(k)}$ if and only if the highest weight of $\pi_0$ is of the form $(s_1', \ldots, s_n'; t_1, \ldots, t_n)$ where

$$s_n' \geqslant k \qquad \text{and} \qquad t_n \geqslant 0. \tag{8.23}$$

EXAMPLE 8.9. Let $\mathbb{F} = \mathbb{H}$. Then $G = \mathcal{O}_*(4n)$. Let $\pi$ be an ifdp representation of $A$ with highest weight $(s_1, \ldots, s_{2n})$. Then by (5.13) and Example 3.5, $\omega(\pi^{(k)}) = s_{2n-1} + s_{2n} + 2k$ and $s_{2n} \geqslant 0$. By (8.19), $\pi_0 = \pi^{(k)}$ indexes a member of the holomorphic discrete series if and only if

$$s_{2n-1} + s_{2n} + 2k > 4n - 3. \tag{8.24}$$

Thus, $T(\cdot, \pi_0)$ appears in $\mathscr{R}^{(k)}$ if and only if the highest weight of $\pi_0$ is of the form $(s_1', \ldots, s_{2n}')$ where

$$s_{2n}' \geqslant k. \tag{8.25}$$

Notice that when $\mathbb{F} = \mathbb{R}$ and $k = 2n$ *all* the holomorphic discrete series representations appear in $\mathcal{R}^{(k)}$. In fact, as we shall briefly indicate, in this situation each representation in the holomorphic discrete series appears in $\mathcal{R}^{(k)}$ with multiplicity $2d_\lambda$ rather than $d_\lambda$. This is the exceptional case corresponding to $\mathbb{R}^{2n\times n}$, and the preceding analysis goes through with minor modifications. Since this example has been treated by Gelbart in [16], we shall be content here with a bare outline. Let $X = \mathbb{R}^{2n\times n}$. Then $\tilde{U}_X = SO(2n)^\wedge$; i.e., every irreducible representation $\lambda$ appears in the $SO(2n)$-primary decomposition of $L^2(X)$. The mapping $\pi \to \lambda = \lambda(\cdot, \pi)$ in this context accounts for only half of $SO(2n)^\wedge$. The remaining half of the dual object consists of the representations $\tilde{\lambda} = \tilde{\lambda}(\cdot, \pi)$ defined by $\tilde{\lambda}(u) = \lambda(\sigma u \sigma^{-1})$ where $\sigma = \operatorname{diag}(1,..., 1, -1)$, an orthogonal matrix of determinant $-1$. Then the spaces $L_\lambda{}^2(X, \mathcal{L}_\lambda)$ and $L_\lambda{}^2(X, \mathcal{L}_{\tilde{\lambda}})$ are unitarily equivalent; the corresponding Bessel functions $J_\lambda$ and $J_{\tilde{\lambda}}$ coincide; and the representations $\mathcal{R}(\cdot, \lambda)$ and $\mathcal{R}(\cdot, \tilde{\lambda})$ (cf., (8.11)) are unitarily equivalent. The following theorem gives the analog for $\mathbb{R}^{2n\times n}$ of Theorem 8.3.

THEOREM 8.10. *The metaplectic representation $\mathcal{R} = \mathcal{R}^{(2n)}$ of $G = Sp(n, \mathbb{R})$ on the space $L^2(\mathbb{R}^{2n\times n})$ decomposes as*

$$\mathcal{R} \cong \sum_\pi \oplus\, 2R(\cdot, \pi, 2n) \tag{8.26}$$

*where the sum extends over all* ifdp *representations $\pi$ of $A$. Furthermore, for $\omega(\pi) > 0$*

$$R(\cdot, \pi, 2n) \cong d_\lambda R(\cdot, \pi^{(2n)}) \tag{8.27}$$

*where the representations $R(\cdot, \pi^{(2n)})$ are irreducible and exhaust the holomorphic discrete series for $Sp(n, \mathbb{R})$. When $\omega(\pi) = 0$, the representations $R(\cdot, \pi^{(2n)})$ can be realized in Hilbert spaces of holomorphic functions on $H$, but these representations are not square-integrable.*

We conclude the paper with some comments on "limits" of holomorphic discrete series. Definition 8.5 describes a collection of Hilbert spaces $\mathfrak{H}_\pi$ of holomorphic functions on $H$ which for the range

$$(n-1)\nu/2 < \omega(\pi) \leqslant m + (n-1)\nu/2 \tag{8.28}$$

inherit an inner product from that of $L_\pi{}^2(A_0)$, but this inner product is not derived from a measure as in (8.15). When $\pi$ is scalar, or more generally when the restriction of $\pi$ to $\mathcal{O}_0$ is irreducible, one can describe these spaces more explicitly in terms of their kernel functions (cf. [2, 21]) and formula (8.16) leads to new representations of $G$ in these spaces.

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